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On the existence of efficient and fair extensions of communication values for connected graphs☆

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HIGHLIGHTS

- We study values for TU games with a communication graph (CO-values).
- We study CO-values for connected graphs that are fair and efficient.
- There exists a unique efficient and fair extension to the whole domain.

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ABSTRACT

We study values for TU games with a communication graph (CO-values). In particular, we show that CO-values for connected graphs that are fair and efficient allow for a unique efficient and fair extension to the full domain.

1. Introduction

Cooperative games with transferable utility, or simply TU-games, are a simple but versatile tool to model the worth generated by coalitions of players. In many situations, the generation of worth also is affected by way the players are organized. Myerson (1977) suggests to represent the relation between players by undirected graphs. A TU-game together with a graph on its player set is called a communication game or CO-game. A communication value or CO-value assigns a payoff to any player of a CO-game.

Myerson (1977) himself introduces a CO-value, nowadays called the Myerson value. A crucial feature of this CO-value is fairness, i.e., removing a link from the graph changes the payoffs

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of the two players involved by the same amount. Another property of this CO-value is component efficiency, i.e., the payoffs of the players of a component of the graph add up to the worth generated by this component. This property indicates that the Myerson value implies the components of the graph to be the productive units. In contrast, the CO-values proposed by Casajus (2007), Hamiache (2012), Béal et al. (2012), and van den Brink et al. (2012) are efficient, i.e., all players' payoffs sum up to the worth of the grand coalition. This indicates that the grand coalition is the productive unit, while the components of the graph represent groups formed in the process of bargaining on the distribution of the grand coalition's worth.

For CO-games with connected graphs (for short, connected games), it is clear that the Myerson value is efficient. Recently, Béal et al. (2015, Theorem 5) show that the CO-value suggested by van den Brink et al. (2012) is the unique efficient and fair extension of the Myerson value for connected games to the full domain of CO-games. That is, their CO-value is the unique fair and efficient CO-value that coincides with the Myerson value for connected games. This result rests on the insight that there exists at most one fair and efficient extension of a CO-value for connected games (Béal et al., 2015, Theorem 4). Of course, such a CO-value itself must be fair and efficient for connected games.

There are many CO-values that are efficient and fair for connected games. So far, it is an open question which of these allow for an efficient and fair extension to the full domain of CO-games. In this paper, we explore this matter. As our main result, we show that all CO-values that are efficient and fair for connected games can be extended in the afore-mentioned sense (Theorem 3).

The article is organized as follows. Section 2 gives basic definitions and notations. The fair and efficient extendability is studied in Section 3. Some remarks conclude the paper. The Appendix contains the proof of our main result.

2. Cooperative games and graphs

Fix a non-empty and finite set of players N. For $C, S, T \subseteq N$ and N, let c, s, t, and n denote their cardinalities. A **TU-game** on N is given by a **coalition function** $v \in \mathbb{V} := \{f : 2^N \longrightarrow \mathbb{R} \mid f(\emptyset) = 0\}$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**, and $v(S)$ is called the worth of coalition S. A **value** on $\mathbb{V}$ is an operator $\varphi : \mathbb{V} \to \mathbb{R}^N$. The **Shapley value** (Shapley, 1953), Sh, is given by

$$\mathrm{Sh}_i(v) = \sum_{S \subseteq N\setminus\{i\}} \frac{s! \cdot (n-1-s)!}{n!} \cdot (v(S \cup \{i\}) - v(S))$$

for all $v \in \mathbb{V}$ and $i \in N$.

A **communication graph** for $S \subseteq N$ is an undirected graph (S, L) given by a link set $L \subseteq L^S := \{\{i,j\} \mid i,j \in S, i \neq j\}$; a typical element (**link**) of L is written as $ij := \{i,j\}$; $\mathcal{L}^S$ denotes the set of all subsets of L^S. For $i \in S$ and $L \in \mathcal{L}^S$, $L_i := \{\lambda \in L \mid i \in \lambda\}$. For $L \in \mathcal{L}^N$, $L|_S := \{\lambda \in L \mid \lambda \subseteq S\}$. The players $i, j \in S$ are called connected in (S, L) if there is a sequence of players $(i_1, i_2, \ldots, i_k)$, $k \in \mathbb{N}$, $k > 1$ from S such that $i_1 = i$, $i_k = j$, and $i_\ell i_{\ell+1} \in L$ for all $\ell \in \{1, \ldots, k-1\}$. It is clear that connectedness is an equivalence relation. Hence, it induces a partition $\mathcal{C}(S, L)$ of S, the set of **components** of (S, L), such that $C \in \mathcal{C}(S, L)$, $i, j \in C$, $k \in S \setminus C$, $i \neq j$ implies that i and j are connected and that i and k are not connected in L. The component of (S, L) containing $i \in S$ is denoted by $C_i(S, L)$. The graph (S, L) is called **connected** if $\mathcal{C}(S, L) = \{S\}$; $\mathcal{L}^S_C \subseteq \mathcal{L}^S$ denotes the set of connected graphs on S.

A **CO-game** is a pair $(v, L) \in \mathcal{G} := \mathbb{V} \times \mathcal{L}^N$. A CO-game is called connected if the associated graph is connected. We denote by $\mathcal{G}_C \subseteq \mathcal{G}$ the class of all **connected CO-games**. A **CO-value** on some class of CO-games $\mathcal{G}^* \subseteq \mathcal{G}$ is an operator $\varphi : \mathcal{G}^* \to \mathbb{R}^N$. The **Myerson value** (Myerson, 1977) is the CO-value on $\mathcal{G}$ given by

$$\mathrm{My}(v, L) := \mathrm{Sh}\left(v^L\right), \qquad v^L(S) := \sum_{T \in \mathcal{C}(S, L|_S)} v(T), \quad S \subseteq N$$

for all $(v, L) \in \mathcal{G}$. It is characterized by component efficiency and fairness below. The **Efficient Egalitarian Myerson value** (van den Brink et al., 2012) is given by

$$\mathrm{eeMy}_i(v, L) := \mathrm{My}_i(v, L) + \frac{v(N) - v^L(N)}{n}$$

for all $(v, L) \in \mathcal{G}$ and $i \in N$.

Throughout this article, we sometimes invoke axioms on subclasses of CO-games indicated by "$|_{\mathcal{G}^*}$", $\mathcal{G}^* \subseteq \mathcal{G}$ in their definition. For any such subclass, all the CO-games used in the axiom belong to the subclass.

Component efficiency, CE$|_{\mathcal{G}^*}$. A value φ satisfies **CE**$|_{\mathcal{G}^*}$ if

$$\sum_{i \in C} \varphi_i(v, L) = v(C) \quad \text{for all } (v, L) \in \mathcal{G}^* \text{ and } C \in \mathcal{C}(N, L).$$

Fairness, F$|_{\mathcal{G}^*}$. A value φ satisfies **F**$|_{\mathcal{G}^*}$ if

$$\varphi_i(v, L \cup \{ij\}) - \varphi_i(v, L) = \varphi_j(v, L \cup \{ij\}) - \varphi_j(v, L)$$

for all $(v, L) \in \mathcal{G}^*$ and $ij \in L^N \setminus L$ such that $(v, L \cup \{ij\}) \in \mathcal{G}^*$.

Efficiency, E$|_{\mathcal{G}^*}$. A value φ satisfies **E**$|_{\mathcal{G}^*}$ if

$$\sum_{i \in N} \varphi_i(v, L) = v(N) \quad \text{for all } (v, L) \in \mathcal{G}^*.$$

3. Existence of efficient and fair extensions

Recently, Béal et al. (2015) established the following results on fair and efficient extensions of CO-values for connected games.

Theorem 1 (*Béal et al., 2015, Theorem 4*). *Let φ and ψ be two CO-values on $\mathcal{G}$ that satisfy efficiency (**E**$|_{\mathcal{G}}$) and fairness (**F**$|_{\mathcal{G}}$). If $\varphi(v, L) = \psi(v, L)$ for all $(v, L) \in \mathcal{G}_C$, then $\varphi = \psi$ for all $(v, L) \in \mathcal{G}$.*

That is, there exists at most one fair and efficient extension of a CO-value for connected games. It is easy to check that the Efficient Egalitarian Myerson satisfies efficiency and fairness and coincides with the Myerson value for connected games. This clarifies existence in the next theorem, which states that the Efficient Egalitarian Myerson value is the unique efficient and fair extension of the Myerson value.

Theorem 2 (*Béal et al., 2015, Theorem 5*). *A value φ satisfies $\varphi = \mathrm{My}$ on $\mathcal{G}_C$ and meets efficiency (**E**$|_{\mathcal{G}}$) and fairness (**F**$|_{\mathcal{G}}$), if and only if $\varphi = \mathrm{eeMy}$ on $\mathcal{G}$.*

Besides the Myerson value, there are many other CO-values on $\mathcal{G}$ that are efficient and fair for connected games, for example, the CO-value $\varphi^\star$ given by

$$\varphi_i^\star(v, L) = v(C) \cdot \left(\frac{|L_i|}{2 \cdot \left|L^C\right|} + \frac{1}{c} - \frac{|L|_C|}{c \cdot \left|L^C\right|}\right)$$

for all $(v, L) \in \mathcal{G}$, $C \in \mathcal{C}(N, L)$, and $i \in C$. For any of her links, a player obtains a fraction of $1/\left(2 \cdot \left|L^C\right|\right)$ of the worth generated by her component. The rest of this component's worth is distributed equally among its members.

In this paper, we tackle the question whether there is an efficient and fair extension of such a communication value. Indeed, the reader may check that the CO-value $\varphi^\heartsuit$ given by

$$\varphi_i^\heartsuit(v, L) = v(N) \cdot \left(\frac{|L_i|}{2 \cdot \left|L^N\right|} + \frac{1}{n} - \frac{|L|}{n \cdot \left|L^N\right|}\right)$$