Economics Letters 143 (2016) 9–12

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

On the (in)stability of nonlinear feedback solutions in a dynamic duopoly with renewable resource exploitation[☆]

Luca Lambertini[*], Andrea Mantovani

Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy

HIGHLIGHTS

- We study linear and nonlinear feedback equilibria in a differential duopoly game with resource extraction.
- We show that the degenerate tangency solution is unstable.
- The continuum of nonlinear equilibria is characterised using Rowat's (2007) procedure.
- The set of stable nonlinear equilibria is also identified.

ARTICLE INFO

Article history:
Received 10 February 2016
Received in revised form
8 March 2016
Accepted 18 March 2016
Available online 22 March 2016

JEL classification:
C73
L13
Q2

Keywords:
Differential games
Renewable resources
Feedback strategies

ABSTRACT

We revisit Fujiwara's (2008) linear–quadratic differential duopoly game to show that the degenerate nonlinear feedback identified by the tangency point with the stationary state line is indeed unstable, given the dynamics of the natural resource exploited by firms. To do so, we fully characterise the continuum of nonlinear feedback solution via Rowat's (2007) method, thereby identifying the infinitely many stable nonlinear feedback equilibria. This entails that Rowat's method can be used in games where each player's instantaneous payoff depends quadratically on all players' controls.

1. Introduction

Differential games are dynamic games in continuous time in which players' strategies affect both their instantaneous payoffs and the stock of at least one relevant state variable by entering the differential equation describing the evolution of the state itself. In turn, the state dynamics may affect players' strategies and objective functions, a fact which the players may or may not account for, depending on the structure of the information underpinning the solution concept being used.

A relevant class of differential games, which lends itself to a fully analytical solution, is that characterised by a linear–quadratic (LQ) structure, in which the payoff function is quadratic in controls (and possibly, but not always, in the state) while the state equation is linear in the state and the controls (see Dockner et al., 2000, Section 7.1). An LQ game can be solved in strategies which are linear or nonlinear in the relevant state variable. The nonlinear approach delivers a continuum of infinitely many solutions. The first notable contribution to the nonlinear solution of LQ games is Tsutsui and Mino (1990), using the sticky price game introduced by Fershtman and Kamien (1987).[1] The drawback of their analysis is the endogenisation of the strategy domain over a subset of the state space, posing boundaries to instantaneous payoffs. A way out lies in the method proposed by Rowat (2007), whereby the characterisation

☆ We would like to thank an anonymous referee for precious comments. The usual disclaimer applies.

* Corresponding author.
E-mail addresses: luca.lambertini@unibo.it (L. Lambertini), a.mantovani@unibo.it (A. Mantovani).

[1] Nonlinear feedback solutions have been subsequently investigated in oligopoly theory, environmental and resource economics and other fields. See Shimomura (1991), Dockner and Sorger (1996), Itaya and Shimomura (2001), Rubio and Casino (2002) and Colombo and Labrecciosa (2015), *inter alia*.

http://dx.doi.org/10.1016/j.econlet.2016.03.015
0165-1765/

of nonlinear strategies can be addressed with unbounded controls and payoffs.

With all this in mind, we revisit Fujiwara's (2008) model of dynamic duopolistic exploitation of a renewable resource, to fully characterise the continuum of nonlinear feedback strategies and show that Fujiwara's claim:

"Rowat (2007, pp. 3193–3194) gives useful conditions for [the nonlinear feedback control] to be an equilibrium strategy. At first sight, they are violated in the present model but such a guess is incorrect since Rowat (2007) assumes that player i's payoff does not depend on its rival's strategy" (Fujiwara, 2008, p. 219)

is indeed mistaken, since replicating Rowat's method in Fujiwara's game allows us (i) to show that the tangency solution is indeed unstable due to the dynamic properties of the model, and (ii) to characterise the continuum of stable nonlinear feedback equilibria identified by the appropriate intersections between isoclines and the steady state locus. Each of the equilibria belonging to this set can be reached provided that the initial stock of the resource is low enough.

Hence, our note illustrates that Rowat's method is applicable to LQ games where all of the players' controls appear in each instantaneous payoff function, including the aforementioned sticky price game, where nonlinear strategies could be characterised anew using Rowat's approach instead of Tsutsui and Mino's (as already anticipated in Rowat, 2007, p. 3182).

The remainder of the paper is organised as follows. Section 2 illustrates the setup. The linear feedback solution is illustrated in Section 3. Nonlinear feedback strategies are dealt with in Section 4. Concluding remarks are in Section 5.

2. The model

The setup is the same as in Benchekroun (2003), Fujiwara (2008) and Colombo and Labrecciosa (2015). The model illustrates a differential oligopoly game of resource extraction unravelling over continuous time $t \in [0, \infty)$. The market is supplied by two fully symmetric firms[2] producing a homogeneous good, whose inverse demand function is $p = a - X$ at any time t, with $X = \sum_{i=1}^{2} x_i$. Firms share the same technology, characterised by marginal cost $c \in (0, a)$, constant over time. The individual instantaneous profit function is $\pi_i = (p - c)\, x_i$. Firms exploit a common pool renewable resource, whose evolution over time is described by the following dynamics:

$$\dot{S} = kS - X \tag{1}$$

where S is the resource stock, $k > 0$ is its time-invariant growth rate.[3]

Firms play noncooperatively and choose their respective outputs simultaneously at every instant. In the remainder, in order to save upon notation, we will pose $\sigma \equiv a - c > 0$. The ith firm maximises the discounted profit flow

$$\Pi_i = \int_0^{\infty} \pi_i e^{-rt} dt = \int_0^{\infty} \left(\sigma - x_i - x_j\right) x_i e^{-rt} dt, \tag{2}$$

under the constraint posed by the state equation (1). The initial condition is $S(0) = S_0 > 0$. Parameter $r > 0$ is the discount rate, common to all firms and constant over time. To guarantee the positivity of the residual resource stock at the steady state under linear feedback strategies, as in Fujiwara (2008, p. 218), the ensuing analysis will be carried out under.

Assumption 1. $k > 5r/2$.

Before delving into calculations, a few words are in order as to the solution concepts and the notion of stability which are about to be used. The simplest solution concept is the open-loop Nash equilibrium, whereby firms do not condition their output strategies on the resource stock at any time, and design their output plans once and for all at $t = 0$. In this scenario, firms play the individual (static) Cournot–Nash output $x^{CN} = \sigma/3$ at all times, and the residual amount of the natural resource in steady state is $S^{CN} = 2\sigma/(3k) = X^{CN}/k$. If instead firms take explicitly into account their impact on the stock at all times, the relevant solution concept is the feedback Nash equilibrium, which is often labelled as Markov perfect equilibrium (e.g., Dockner et al., 2000, p. 102 and Rowat, 2007, p. 3186). Since the present setup identifies a linear–quadratic differential game, feedback information gives rise to strategies which can be either linear or nonlinear in the state variable (for more, see Dockner et al., 2000, Section 7.1). Moreover, since here the instantaneous payoff is quadratic in controls and the state equation is linear both in the state and in the vector of controls, the present game is a linear state one and the open-loop solution is a degenerate feedback one (see Dockner et al., 2000, Section 7.2). In the remainder, the stability or instability of any given solution is revealed by the dynamics of the state, i.e., by the sign of (1) above and below the locus $\dot{S} = 0$ univocally identified by $S = X/k$.[4]

3. The linear feedback solution

Here, we follow the undetermined coefficient technique (Dockner et al., 2000, pp. 180–83). The Hamilton–Jacobi–Bellman (HJB) of firm i is

$$rV_i(S) = \max_{x_i} \left[(\sigma - X)\, x_i + V_i'(S)\,(kS - X) \right] \tag{3}$$

where $V_i(S)$ is firm i's value function, and $V_i'(S) = \partial V_i(S)/\partial S$. The first order condition (FOC) on x_i is

$$\sigma - 2x_i - x_j - V_i'(S) = 0 \tag{4}$$

which must hold together with terminal condition $\lim_{t\to\infty} e^{-rt} V(s) = 0$ (as in Fujiwara, 2008, p. 218). In view of the *ex ante* symmetry across firms, we impose the symmetry conditions $x_i = x(S)$ and $V_i(S) = V(S)$ for all i and solve FOC (4) to obtain

$$x^F(S) = \max\left\{0, \frac{\sigma - V'(S)}{3}\right\} \tag{5}$$

---

2 The analysis of oligopolistic interaction in the same setup is in Benchekroun (2008) and Lambertini and Mantovani (2014).

3 From Benchekroun (2003) onwards, the resource dynamics is defined as

$$\dot{S} = F(S) - X$$

with

$$F(S) = \begin{cases} kS & \forall S \in (0, S_y] \\ kS_y \left(\dfrac{S_{\max} - S}{S_{\max} - S_y}\right) & \forall S \in (S_y, S_{\max}] \end{cases}$$

where k is the *implicit* growth rate when the stock is at most equal to S_y and kS_y is the maximum sustainable yield. This formulation implies that (i) if the resource stock is sufficiently small the population grows at an exponential rate; and (ii) beyond S_y, the asset grows at a decreasing rate. Moreover, $S_{\max}$ is the *carrying capacity* of the habitat, beyond which the growth rate of the resource is negative, being limited by available amounts of food and space.

4 The standard procedure consists in evaluating the sign of the eigenvalues of the Jacobian matrix of the state-control system; or, since the present setup generates a 2 × 2 Jacobian matrix, the sign of its trace and determinant (see Dockner et al., 2000, pp. 69–70 and 256–57).