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Ordinally consistent set-ranking methods for tournaments

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HIGHLIGHTS

- We study methods for ranking sets of items in a tournament.
- We define a notion of ordinal consistency for such methods.
- We construct two examples of ordinally consistent set-ranking methods.

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ABSTRACT

A set-ranking method assigns to each tournament on a given set an ordering of the subsets of that set. Such a method is consistent if (i) the items in the set are ranked in the same order as the sets of items they beat and (ii) the ordering of the items fully determines the ordering of the sets of items. We describe two consistent set-ranking methods.

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1. Introduction

We reconsider the problem of extracting an ordering from a tournament. If the incidence matrix of a tournament on m items is irreducible, the Perron–Frobenius theorem ensures that it possesses a unique eigenvector in the m-simplex. The eigenvector solution assigns to each item x a rating equal to the value of the xth coordinate of that eigenvector (Landau, 1895; Wei, 1952; Kendall, 1955). The rating of x is thus proportional to the sum of the ratings of the items that x beats in the tournament. This self-consistency property is what lends appeal to the solution.

Implicitly, the eigenvector solution defines what may be called a *set-rating method*. It assigns a rating not only to each item but also to each set of items: the rating of a set is the sum of the ratings of its members.

Of course, as a by-product, the solution delivers a *ranking* of the sets of items—a set is ranked above another if and only if its rating is higher. But the construction of this ranking (hence also the construction of the ranking of items it induces) requires that the strength of an item be *cardinally* measurable. Indeed, the condition that an item's rating be *proportional* to the rating of the set it beats is based on that assumption. Moreover, if the items' ratings have no cardinal meaning, the ordering of two sets of items should not vary with an increasing transformation of the ratings of their members—but it clearly does since it depends upon the *sum* of these ratings. The eigenvector solution is inherently cardinal.

This note formulates a version of the consistency property of the eigenvector solution that does not assume cardinal measurability of the strength of the items. We call a set-ranking method consistent if (i) the items are ranked in the same order as the sets they beat and (ii) the ordering of the items completely determines the ordering of the sets of items. While the eigenvector set-ranking method satisfies the first condition, it violates the second. The question arises whether these conditions are compatible. We prove that they are, and describe two consistent set-ranking solutions.

Our work is related to the decision-theoretic literature on choices of "menus", i.e., *sets* of items. Kreps (1979) was the first to observe that in a dynamic environment where choices are made in several stages, early choices amount to choices of menus. This led to the development of a decision-theoretic literature axiomatizing various classes of individual preferences over menus: see Barberà et al. (2004) for a survey. This literature proved useful to analyze time-inconsistent behavior exhibiting, for instance, preference for

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flexibility (as in Kreps, 1979) or for commitment (as in Gul and Pesendorfer, 2001).

Groups, just like individuals, make dynamic choices. But the preferences of a democratic society over (social) items – as expressed by the majority relation generated by the preferences of its members – are typically not transitive: indeed, McGarvey's (1953) theorem tells us that they can be represented by an arbitrary tournament. In a dynamic context where society chooses menus of social items, it is therefore important to extract from the majority tournament over items a social preference relation over sets of items. To guarantee coherent social choices, this relation should be an ordering. Set-ranking methods for tournaments are procedures to construct such orderings.

2. Definitions

Let X be a finite set of m items and let $\mathcal{X}$ be the set of nonempty subsets of X. A tournament is a complete and asymmetric binary relation T on X. Let $\mathcal{T}$ denote the set of tournaments. If $T \in \mathcal{T}$ and $x \in X$, let $t(x) = \{y \in X : xTy\}$. Let $\mathcal{R}(X)$ be the set of orderings of X and let $\mathcal{R}(\mathcal{X})$ be the set of orderings of $\mathcal{X}$.

A *set-ranking method* is a function $R : \mathcal{T} \to \mathcal{R}(\mathcal{X})$. We interpret $R(T)$ as the ordering of $\mathcal{X}$ recommended by the method R for the tournament T. Let $P(T)$ and $I(T)$ denote, respectively, the strict ordering and the equivalence relation generated by the ordering $R(T)$. Denote by $R_X(T) \in \mathcal{R}(X)$ the ordering of the items induced by $R(T)$: by definition, $xR_X(T)y$ if and only if $\{x\}\,R(T)\,\{y\}$. We call $R_X : \mathcal{T} \to \mathcal{R}(X)$ a *ranking method*.

A set-ranking method R is *consistent* if it satisfies the following two conditions:

(i) for all $T \in \mathcal{T}$ and $x, y \in X$, $xR_X(T)y \Leftrightarrow t(x)R(T)t(y)$,
(ii) for all $T, T' \in \mathcal{T}$, $R_X(T) = R_X(T') \Rightarrow R(T) = R(T')$.

The first condition says that the ranking of two items should be the same as the ranking of the sets they beat: item x is stronger than y if and only if x beats a stronger set than y does. This is the ordinal version of the self-consistency property of the eigenvector solution. The second condition says that the ranking of the items fully determines the ranking of the sets of items: the *extension rule* for deriving an ordering on $\mathcal{X}$ from one on X is the same in every tournament. The eigenvector solution imposes a cardinal version of this requirement: the rule for extending ratings from items to sets does not vary with T—moreover, it takes the particular form of the summation. Note that in the absence of condition (ii), condition (i) has no bite: the partial ordering on $\mathcal{X}$ derived from $R_X(T)$ and condition (i) can always be completed.

Condition (i) imposes severe restrictions on the extension procedure in condition (ii). We describe two examples of consistent set-ranking methods. Characterizing the set of consistent methods is an open problem.

3. Results

A tournament is irreducible if its transitive closure is a complete relation. Every tournament can be decomposed into a collection of uniquely defined irreducible components: the top component is the top cycle, the second is the top cycle of the restriction of the tournament to the remaining items, and so on. The *decomposition ordering* ranks the items according to the irreducible component they belong to.

Formally, for any ordering $R_0 \in \mathcal{R}(X)$ and $Y \in \mathcal{X}$, let $\max_Y R_0$ denote the set of maximal elements of R_0 in Y. Since yI_0y' for all $y, y' \in \max_Y R_0$, we abuse notation and write $(\max_Y R_0)R_0(\max_Z R_0)$ if yR_0z for all $y \in \max_Y R_0$ and $z \in \max_Z R_0$. The *top cycle* of a tournament T is the set $X_1(T) := \max_X \overline{T}$ of maximal elements of the transitive closure $\overline{T}$ of T in X. For any $Y \in \mathcal{X}$, let T_Y denote the restriction of tournament T to the subset of items Y. Define inductively $X_k(T)$ to be the set of maximal elements of $\overline{T_{X\setminus\cup_{h=1}^k X_h(T)}}$ in $X \setminus \cup_{h=1}^k X_h(T)$. The resulting partition $\{X_1(T), \ldots, X_K(T)\}$ of X defines the *decomposition ordering* $R_X^*(T)$ of X:

$$xR_X^*(T)y \Leftrightarrow k(x, T) \le k(y, T),$$

where $k(z, T)$ is the unique integer k such that $z \in X_k(T)$.

Call a set-ranking method R' *finer* than R if for all $T \in \mathcal{T}$ and all $Y, Z \in \mathcal{X}$, $YP(T)Z \Rightarrow YP'(T)Z$.

Proposition 1. *There exists a unique finest consistent set-ranking method R such that*

$$YR(T)Z \Leftrightarrow (\max_Y R_X(T))R_X(T)(\max_Y R_X(T)) \tag{1}$$

for all $T \in \mathcal{T}$ and $Y, Z \in \mathcal{X}$. The induced ranking method R_X chooses the decomposition ordering of X in each $T \in \mathcal{T}$.

Like the eigenvector method, the set-ranking method in Proposition 1 ranks items according to the strength of the set of items they beat—it satisfies condition (i) in the definition of Consistency. But the method ranks sets of items according to the strength of their *strongest* member, not according to the sum of the strengths of their members. This ensures that it satisfies condition (ii) in the definition of Consistency, contrary to the eigenvector method.

Proof of Proposition 1. For every $a \in \{0, 1, \ldots, m-1\}^X$, define the ordering $R^a \in \mathcal{R}(\mathcal{X})$ by

$$YR^aZ \Leftrightarrow \max_{y\in Y} a_y \ge \max_{z\in Z} a_z. \tag{2}$$

Call $a, a' \in \{0, 1, \ldots, m-1\}^X$ *ordinally equivalent* if they generate the same ordering, that is, $R^a = R^{a'}$. Call them *ordinally compatible* if they generate compatible orderings:

$$YP^aZ \Rightarrow YR^{a'}Z \quad \text{and} \quad YP^{a'}Z \Rightarrow YR^aZ. \tag{3}$$

Call a' *finer* than a if for all $Y, Z \in \mathcal{X}$, $YP^aZ \Rightarrow YP^{a'}Z$.

For any $T \in \mathcal{T}$, define the function $f^T : \{0, 1, \ldots, m-1\}^X \to \{0, 1, \ldots, m-1\}^X$ by

$$f_x^T(a) = \max_{y\in t(x)} a_y \quad \text{for all } x \in X,$$

where, by convention, $\max_{y\in\emptyset} a_y = 0$. Since $\{0, 1, \ldots, m-1\}^X$ is a complete lattice and f^T is nondecreasing, Tarski's theorem implies that f^T has a fixed point: there exists $a \in \{0, 1, \ldots, m-1\}^X$ such that

$$a_x = \max_{y\in t(x)} a_y \quad \text{for all } x \in X.$$

We claim that all fixed points of f^T are ordinally compatible. To see why, let a, a' be two such fixed points and check first that for any $x, y \in X$,

$$a_x > a_y \Rightarrow a'_x \ge a'_y \quad \text{and} \quad a'_x > a'_y \Rightarrow a_x \ge a_y. \tag{4}$$

If, say, $a_x > a_y$ and $a'_x < a'_y$, then $\max_{z\in t(x)} a_z > \max_{z\in t(y)} a_z$ and $\max_{z\in t(x)} a'_z < \max_{z\in t(y)} a'_z$. But either xTy or yTx. If xTy, then $y \in t(x)$ and

$$a'_y \le \max_{z\in t(x)} a'_z < \max_{z\in t(y)} a'_z = a'_y,$$

a contradiction. If yTx, a similar contradiction arises. Statements (4) and (2) now imply (3), i.e., a, a' are ordinally compatible.

It follows that the finest fixed points of f^T are all ordinally equivalent. Call $R(T)$ the common ordering they induce on $\mathcal{X}$ through (2). By construction, R is consistent, and it is the finest consistent set-ranking method satisfying (1). That $R_X(T)$ coincides with the decomposition ordering of X at T is a matter of checking. ■

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