Economics Letters 145 (2016) 168–171

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

On the consistency of random serial dictatorship☆

Xiang Han

Department of Economics, Southern Methodist University, 3300 Dyer Street Suite 301, Dallas, TX 75275, United States

ARTICLE INFO

Article history:
Received 12 April 2016
Accepted 3 June 2016
Available online 16 June 2016

JEL classification:
D61
D63
D70
C78

Keywords:
Consistency
Random serial dictatorship
Random assignment
Strategy-proofness

ABSTRACT

The random serial dictatorship (RSD) can be generalized to indivisible object allocation problems allowing fractional endowments such that symmetry, ex-post efficiency and strategy-proofness are preserved. However, there exists a consistent extension of RSD if and only if the population is less than four. The inconsistency of the generalized RSD is a common feature of strategy-proof rules that satisfy minimal fairness and efficiency properties: symmetry, ex-post efficiency, consistency and strategy-proofness are not compatible.

1. Introduction

When two people both demand a single indivisible object, flipping a coin or playing rock–paper–scissors is a common fair solution. This is the simplest form of *random serial dictatorship* (RSD). Generally, in a *house allocation* problem (Hylland and Zeckhauser, 1979) a group of n agents collectively own n indivisible objects, and each agent has to be assigned one object without monetary transfer. RSD selects a random assignment by picking an ordering of the agents from the uniform distribution and letting the agents choose their favorite available object sequentially according to this ordering.[1] It satisfies a set of desirable properties: symmetry, ex-post efficiency and strategy-proofness. Incentive compatibility is the main advantage of RSD over another solution to the random assignment problem: the *probabilistic serial rule* (PS) (Bogomolnaia and Moulin, 2001). In PS, the agents consume the probability shares of their best available objects simultaneously at the unit rate. While PS is not strategy-proof, it satisfies stronger efficiency and fairness properties: sd-efficiency and sd-no-envy. Moreover, it is consistent (Thomson, 2010; Heo, 2014). Then a natural question is that whether RSD also satisfies the consistency principle.

Consistency is a robustness or stability concept that requires a rule to be coherent in selecting assignments for any problem and its subproblems.[2] In the context of random assignments, it states the following: suppose the rule recommends some random assignment for a problem, if some agents leave the problem with their probability shares of the objects, then the rule should recommend the same assignment for each agent in the reduced problem as in the original problem.[3] To discuss this axiom, we consider an environment where there is a set of potential agents (the population) and a set of potential objects. In each allocation problem a subset of agents collectively own some probability shares of the objects that sum to the number of these agents. One way to think of the fractional endowments is that there is some uncertainty about the available resources. Another interpretation is that the indivisible objects can be consumed at different times

☆ I am grateful to Rajat Deb for his support and encouragement, to Bo Chen for helpful comments.

E-mail address: xiangh@smu.edu.

1 See Abdulkadiroğlu and Sönmez (1998) for a discussion of RSD and its equivalence to another rule that selects the core from random endowments.

2 See Thomson (2011) for a survey on the consistency principle, and Thomson (2012) for other interpretations of consistency. For studies on consistent deterministic rules, see Ergin (2000), Ehlers and Klaus (2007) and Velez (2014).

3 Chambers (2004) defines another consistency concept, *probabilistic consistency*, for random assignments by considering the departure of some agents with sure objects after their lotteries are realized. He shows that the uniform rule is the only symmetric rule that satisfies probabilistic consistency.

http://dx.doi.org/10.1016/j.econlet.2016.06.005
0165-1765/

and some objects are only available for use during a certain time period.[4]

A rule is an extension of RSD to this general environment if it is in agreement with RSD for each allocation problem with integer endowments.[5] The *generalized RSD* naturally extends RSD by asking the agents to choose their best lottery from the available resources sequentially with respect to each possible ordering. While symmetry, ex-post efficiency and strategy-proofness are preserved, the generalized RSD is not consistent. In fact, we show that if the population is greater than three, then there does not exist any consistent extension of RSD. When the population is three, a consistent extension of RSD can be constructed, but there does not exist any strategy-proof and consistent extension.

Bogomolnaia and Moulin (2001) establish two important impossibility results for house allocation problems: sd-no-envy, ex-post efficiency and strategy-proofness are not compatible; symmetry, sd-efficiency and strategy-proofness are not compatible. Thus if we restrict attention to the class of rules that satisfy the weak fairness and efficiency properties (symmetry and ex-post efficiency, respectively), then strategy-proofness is not compatible with either sd-efficiency or sd-no-envy. We show that there is also a tension between consistency and strategy-proofness: there does not exist a symmetric, ex-post efficient, consistent and strategy-proof rule. Therefore, comparing PS and the generalized RSD in the general allocation problems with fractional endowments, the inconsistency of the latter is also a cost of strategy-proofness.

2. Preliminaries

Let $\mathscr{N}$ denote a finite set of potential agents (the population), and $\mathscr{O}$ a finite set of potential indivisible objects. Assume $|\mathscr{O}| \geq |\mathscr{N}| \geq 3$. Given any $O \subseteq \mathscr{O}$, each agent $i \in \mathscr{N}$ has a complete, transitive and antisymmetric **preference relation** R_i on O. Let $\mathscr{R}_O$ be the set of preference relations on O. $\omega = (\omega_a)_{a\in\mathscr{O}}$ with $\omega_a \in [0, 1]$ for all $a \in \mathscr{O}$ is an **endowment** vector. Denote $\mathscr{S}(\omega) = \{a \in \mathscr{O} : \omega_a > 0\}$. Then a **problem** is a triple $e = (N, \omega, R)$, where $N \subseteq \mathscr{N}$, $\sum_{a\in\mathscr{O}} \omega_a = |N|$,[6] and $R = (R_i)_{i\in N} \in \mathscr{R}^N_{\mathscr{S}(\omega)}$. Let $\mathscr{E}$ be the set of all the problems and $\mathscr{E}^H$ be the set of problems with integer endowments (the *house allocation* problems).

Given $e = (N, \omega, R)$, a **random assignment** is a stochastic matrix $\pi = [\pi_{ia}]_{i\in N, a\in\mathscr{O}}$ such that $\pi_{ia} \geq 0$ for all $i \in N, a \in \mathscr{O}$, $\sum_{a\in\mathscr{O}} \pi_{ia} = 1$ for each $i \in N$ and $\sum_{i\in N} \pi_i = \omega$, where π_i denotes the lottery obtained by agent i. A **deterministic assignment** is a one-to-one function $\mu : N \to \mathscr{S}(\omega)$.[7] By the classical *Birkhoff–Von Neumann Theorem* (Birkhoff, 1946; Von Neumann, 1953), every random assignment can be represented as a lottery over deterministic assignments. A deterministic assignment μ is efficient if it is not Pareto dominated by another deterministic assignment $\mu' : N \to \mu(N)$. π is **ex-post efficient** if it can be represented as a lottery over some efficient deterministic assignments. An agent may be able to compare two lotteries over the objects by the first-order stochastic dominance relation R_i^{sd}: $\pi_i R_i^{sd} \pi_i'$ if $\sum_{b\in\mathscr{S}(\omega):bR_ia} \pi_{ib} \geq \sum_{b\in\mathscr{S}(\omega):bR_ia} \pi'_{ib}$, $\forall a \in \mathscr{S}(\omega)$. Then π is **sd-efficient** if there does not exist π' such that $\pi' \neq \pi$ and $\pi_i' R_i^{sd} \pi_i$, $\forall i \in N$. π is **symmetric** if $R_i = R_j$ implies $\pi_i = \pi_j$ for all $i, j \in N$. π satisfies **sd-no-envy** if $\pi_i R_i^{sd} \pi_j$ for all $i, j \in N$.

A **rule** is a function f that maps each $e \in \mathscr{E}$ to a random assignment $f(e)$. f is said to satisfy a certain property if $f(e)$ satisfies this property for all $e \in \mathscr{E}$. f is **strategy-proof** if for any $e = (N, \omega, R)$, $i \in N$ and $R_i' \in \mathscr{R}_{\mathscr{S}(\omega)}$, $f_i(e) R_i^{sd} f_i(N, \omega, (R_i', R_{-i}))$. Given $e = (N, \omega, R)$, the reduced problem with respect to a random assignment π of e and a subset of agents $I \subseteq N$ is defined as $r_I^\pi(e) = (I, \omega' = \sum_{i\in I} \pi_i, R|_{I,\mathscr{S}(\omega')})$, where $R|_{I,\mathscr{S}(\omega')}$ denotes the restriction of R to the agents I and the objects $\mathscr{S}(\omega')$. Then f is **consistent** if $(f_i(e))_{i\in I} = f(r_I^{f(e)}(e))$ for any $e = (N, \omega, R)$ and $I \subseteq N$.

3. Main results

The *random serial dictatorship* (RSD) is defined on $\mathscr{E}^H$. Given $e \in \mathscr{E}^H$, it selects the random assignment $RSD(e)$ by picking an ordering of the agents from the uniform distribution, and letting the agents choose their best available object sequentially according to this ordering. A rule f is an **extension** of RSD if $f(e) = RSD(e)$ for all $e \in \mathscr{E}^H$. One natural way of extending RSD is to randomize over the *generalized serial dictatorships*. Given any available resources represented by an endowment vector ω with $\sum_{a\in\mathscr{O}} \omega_a \geq 1$, let $\mathscr{L} = \{L \in \Delta\mathscr{S}(\omega) : L_a \leq \omega_a, \forall a \in \mathscr{S}(\omega)\}$ be the set of lotteries that can be picked from ω. It can be easily seen that for any agent i with $R_i \in \mathscr{R}_{\mathscr{S}(\omega)}$, there exists a unique greatest element of $\mathscr{L}$ with respect to R_i^{sd}: any agent can find the "best" lottery from the available resources, which first-order stochastically dominates any other feasible lottery. Then for any $e \in \mathscr{E}$, a generalized serial dictatorship asks the agents to choose their best available lottery sequentially according to an ordering, and the **generalized RSD** simply assigns equal probabilities to each possible generalized serial dictatorship.[8]

Proposition 1. *The generalized RSD is symmetric, ex-post efficient and strategy-proof.*

The symmetry and strategy-proofness of the generalized RSD are obvious. Ex-post efficiency follows from the fact that any generalized serial dictatorship is sd-efficient and a randomization over sd-efficient random assignments is ex-post efficient. While the desirable properties of RSD can be preserved under such an extension, consistency cannot be achieved.

Example 1. Suppose that $N = \{i, j, k\} \subseteq \mathscr{N}$, $\{a, b, c\} \subseteq \mathscr{O}$, $\omega_a = \omega_b = \omega_c = 1$. The preferences R and $RSD(e = (N, \omega, R))$ are given as follows.

R_i	R_j	R_k
a	b	b
b	a	c
c	c	a

	a	b	c
i	$\frac{5}{6}$	0	$\frac{1}{6}$
j	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
k	0	$\frac{1}{2}$	$\frac{1}{2}$

Suppose that agent k leaves the problem with her assignment. For the reduced problem $e' = (\{i, j\}, (\omega_a = 1, \omega_b = \frac{1}{2}, \omega_c = \frac{1}{2}), (R_i, R_j))$, the generalized RSD selects the following random assignment:

[4] Athanassoglou and Sethuraman (2011) consider the allocation problems with *private* fractional endowments, for which individual rationality is a main concern.

[5] On the other hand, PS is already well defined in this environment.

[6] All the results in this paper still hold in a more general setup where the sum of endowments may be different from the number of agents.

[7] Note that each deterministic assignment can be written as a stochastic matrix, but due to the fractional endowments it might not be a feasible random assignment.

[8] One intuitive interpretation of a generalized serial dictatorship is that the agents consume the probability shares of the objects as in the probabilistic serial rule, but in a sequential manner.