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# Differentially monotonic redistribution of income[☆]

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## HIGHLIGHTS

- We suggest a differential version of monotonicity for redistribution rules.
- Increasing income differentials entail increasing post-redistribution differentials.
- Differential monotonicity implies a flat tax combined with a basic income.

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## ABSTRACT

We suggest a differential version of monotonicity for redistribution rules: whenever the differential of two persons' income weakly increases, then the differential of their post-redistribution rewards also weakly increases. Together with efficiency and non-negativity, differential monotonicity characterizes redistribution via taxation at a fixed rate and equal distribution of the total tax revenue.



## 1. Introduction

In modern societies, their members' income is redistributed via various channels. A simple model to study redistribution of income in an $n$-member society is the following: Its members are numbered from 1 to $n$; $\mathbb{N}_n := \{1, \ldots, n\}$. Redistribution is modeled by mappings (redistribution rules) $f : \mathbb{R}^n_+ \to \mathbb{R}^n$.[1] For $x \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$, $f_i(x)$ denotes member $i$'s income after redistribution.

Monotonicity principles or properties have a long tradition and are ubiquitous in economics and game theory (see e.g. Sprumont, 2008). In this note, we suggest and advocate a new monotonicity property for income redistribution rules—*differential monotonicity*: whenever the differential of two members' income weakly increases, then the differential of their post-redistribution rewards does not decrease. Other than its non-differential cousin *strong monotonicity*, it leaves some room for real redistribution (Theorems 1 and 4). Strong monotonicity: whenever a member's income weakly increases, then her post-redistribution reward also weakly increases.

We first show that differential monotonicity is a tuned up version of the order preservation property (Theorem 2). Order preservation: a member with a weakly higher income than another one obtains a weakly higher post-redistribution reward than this other member. Second, we decompose differential monotonicity into a much weaker differential monotonicity property and a strong differential invariance property (Theorem 3).

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[1] Throughout, we disregard the trivial case $n = 1$. Further, we set $\mathbb{R}_+ := [0, +\infty)$.

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In our main result, we make use of differential monotonicity in order to characterize uniform proportional taxation, i.e., taxation by a tax rate that neither depends on individual income nor on the total income of the society, which is combined with an equal distribution of the total tax revenue among the members of the society (Theorem 4). Besides differential monotonicity, we employ two standard axioms, efficiency and non-negativity. Efficiency: total income before redistribution equals total reward after redistribution. Non-negativity: individual rewards after redistribution are non-negative. Note that our main result provides some support the idea of a flat tax combined with an unconditional basic income that depends on the total productivity of the society as suggested by Milner (1920), for example.[2]

The next section gives a formal account and discussion of these results. Some remarks conclude the paper. Appendix A contains the lengthier proof of our main result.

## 2. Differentially monotonic redistribution rules and uniform proportional taxation

We first show that a natural, but strong monotonicity property for redistribution rules essentially prevents real redistribution. In order to avoid this peculiarity, we suggest a differential version of strong monotonicity and explore its basic properties and its relation to some standard properties. Finally, we show that differential monotonicity together with efficiency and non-negativity characterizes redistribution by uniform proportional taxation combined with equal distribution of the total tax revenue.

### 2.1. Strong monotonicity and incentives

The strong monotonicity property for redistribution rules below precludes adverse incentives to earn income, which may be viewed as desirable. When combined with rather innocuous standard properties as efficiency and non-negativity, however, strong monotonicity implies that the members of the society just keep their income. That is, strong monotonicity essentially rules out any kind of real redistribution or solidarity within the society.

**Strong monotonicity, $\mathbf{M}^+$.** For all $x, y \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$ such that $x_i \leq y_i$, we have $f_i(x) \leq f_i(y)$ .[3]

Strong monotonicity requires a non-decreasing individual income to translate into a non-decreasing individual post-redistribution reward. This property is strong because its implication holds irrespective of how the other members' income changes. This implies that increasing ones income at the expense of other members' income is not discouraged.

Let $\mathbf{0} \in \mathbb{R}^n_+$ be given by $\mathbf{0}_\ell = 0$ for all $\ell \in \mathbb{N}_n$. For $i \in \mathbb{N}_n$, $e^i \in \mathbb{R}^n_+$ is given by $e^i_i = 1$ and $e^i_\ell = 0$ for all $\ell \in \mathbb{N}_n \setminus \{i\}$ .

**Efficiency, E.** For all $x \in \mathbb{R}^n_+$, we have $\sum_{\ell \in \mathbb{N}_n} f_\ell(x) = \sum_{\ell \in \mathbb{N}_n} x_\ell$.

The very idea of *re*-distribution suggests that the total rewards after redistribution should not be greater than total income before. In addition, efficiency requires that redistribution has no cost.

**Non-negativity, NN.** For all $x \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$, we have $f_i(x) \geq 0$.

For non-negative income, non-negativity is a very natural requirement. No member of the society necessarily must end up with a negative post-redistribution reward.

[2] The flat tax (or proportional tax) has been advocated in 1845 by McCulloch (1975) and later on by notable others as Mill (1848), Hayek (1960), and Friedman (1962), more recently by Hall and Rabushka (1985) and Hall (1996). Vanderborght and Van Parijs (2005) provide a survey on the idea of an unconditional basic income.

[3] Moulin (1985) suggests a weak version of strong monotonicity called the no-disposal-of-utility property: For all $x, y \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$ such that $x_i \leq y_i$ and $x_\ell = y_\ell$ for all $\ell \in \mathbb{N}_n \setminus \{i\}$, we have $f_i(x) \leq f_i(y)$ .

**Theorem 1.** *A redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies efficiency (**E**), non-negativity (**NN**), and strong monotonicity ($\mathbf{M}^+$) if and only if $f(x) = x$ for all $x \in \mathbb{R}^n_+$.*

**Proof.** It is immediate that the redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ given by $f(x) = x$ for all $x \in \mathbb{R}^n$ meets **E**, **NN**, and $\mathbf{M}^+$. Let now the redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfy **E**, **NN**, and $\mathbf{M}^+$. (i) **E** and **NN** imply $f(\mathbf{0}) = \mathbf{0}$. (ii) By (i) and $\mathbf{M}^+$, we have $f_i(x) = 0$ for all $x \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$ such that $x_i = 0$. (iii) By $\mathbf{M}^+$, we also have $f_i(x) = f_i(y)$ for all $x, y \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$ such that $x_i = y_i$ and $y_\ell = 0$ for all $\ell \in \mathbb{N}_n \setminus \{i\}$ . (iv) By (ii) and **E**, we have $f_i(y) = x_i$. By (iii) and (iv), we finally have $f(x) = x$ for all $x \in \mathbb{R}^n_+$. □

### 2.2. Differential monotonicity

In order to allow for real redistribution without setting adverse incentives to earn income, we suggest and advocate a differential version of strong monotonicity.

**Differential monotonicity, DM.** For all $x, y \in \mathbb{R}^n_+$ and $i, j \in \mathbb{N}_n$, $i \neq j$ such that $x_i - x_j \leq y_i - y_j$, we have $f_i(x) - f_j(x) \leq f_i(y) - f_j(y)$ .

This property demands non-decreasing income differentials of two members to translate into non-decreasing differentials of their post-redistribution rewards. As we will see in the next subsection, differential monotonicity still avoids adverse incentives with respect to increasing ones income but without preventing real redistribution.

In the following, we explore the relation of differential monotonicity to some standard properties of redistribution rules.

**Order preservation, OP.** For all $x \in \mathbb{R}^n_+$ and $i, j \in \mathbb{N}_n$, $i \neq j$ such that $x_i \leq x_j$, we have $f_i(x) \leq f_j(x)$ .

This property guarantees that a member with a weakly higher income ends up with a weakly higher post-redistribution reward.

**Additivity, A.** For all $x, y \in \mathbb{R}^n_+$, we have $f(x + y) = f(x) + f(y)$ .

**Nullity, NY.** We have $f(\mathbf{0}) = \mathbf{0}$.

Since nullity is a rather weak requirement that is implied by additivity and cum grano salis, the theorem below says that differential monotonicity is a tuned up version of the order-preservation property. In particular, differential monotonicity coincides with the order-preservation property in presence of additivity.

**Theorem 2.** *(i) If a redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}$ satisfies the order preservation property (**OP**) and additivity (**A**), then $f$ obeys differential monotonicity (**DM**).*

*(ii) If a redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}$ satisfies differential monotonicity (**DM**) and nullity (**NY**), then $f$ obeys the order preservation property (**OP**).*

**Proof.** (i) Let the redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}$ satisfy **OP** and **A**. Further, let $x, y \in \mathbb{R}^n_+$ and $i, j \in \mathbb{N}_n$, $i \neq j$ be such that (*) $x_i - x_j \leq y_i - y_j$.

W.l.o.g., $0 \leq x_i - x_j$. Set (**) $x^* := x - (x_j - x_i) \cdot e^j$ and (***) $y^* := y - (y_j - y_i) \cdot e^j$. Note that $x^*, y^* \in \mathbb{R}^n_+$ and $x^*_i = x^*_j$ and $y^*_i = y^*_j$. We have

$$\begin{aligned} f_j(x) - f_i(x) &\overset{\mathbf{A}}{=} f_j(x^*) - f_i(x^*) + f_j\left(\left(x_j - x_i\right) \cdot e^j\right) \\ &\quad - f_i\left(\left(x_j - x_i\right) \cdot e^j\right) \\ &\overset{(**),\mathbf{OP}}{=} f_j\left(\left(x_j - x_i\right) \cdot e^j\right) - f_i\left(\left(x_j - x_i\right) \cdot e^j\right) \\ &\overset{(*),\mathbf{OP}}{\leq} f_j\left(\left(x_j - x_i\right) \cdot e^j\right) - f_i\left(\left(x_j - x_i\right) \cdot e^j\right) \\ &\quad + f_j\left(\left(\left(y_j - y_i\right) - \left(x_j - x_i\right)\right) \cdot e^j\right) \\ &\quad - f_i\left(\left(\left(y_j - y_i\right) - \left(x_j - x_i\right)\right) \cdot e^j\right) \end{aligned}$$