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Joint hypothesis tests for multidimensional inequality indices

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HIGHLIGHTS

- We study attribute decomposable multidimensional inequality indices.
- We exploit their decomposition by attributes in several ways.
- We derive the asymptotic distribution of the vector of inequality measures.
- We derive joint hypotheses tests on the vector of inequality measures.
- We present Monte Carlo evidence on their finite sample behavior.

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ABSTRACT

An inequality index over p dimensions of well-being is decomposable by attributes if it can expressed as a function of p unidimensional inequality indices and a measure of association between the various dimensions of well-being. We exploit this decomposition framework to derive joint hypothesis tests regarding the sources of multidimensional inequality, and present Monte Carlo evidence on their finite sample behavior.

Economists often study inequality in well-being using indices that satisfy known sets of ethical axioms and/or statistical properties. The use of unidimensional inequality indices in the analysis of income distributions is well-established (e.g. Cowell and Flachaire, 2007). Recently, multidimensional indices that measure inequality in the joint distribution of several attributes (e.g. income and health) have also received attention, in recognition of well-being as a multidimensional concept (e.g. Justino et al., 2004; Zhong, 2009; Abu-Zaineh and Abul Naga, 2013).

There is an important class of multidimensional inequality indices that satisfy attribute decomposability (Abul Naga and Geoffard, 2006; Kobus, 2012). This property allows a multidimensional index to be disaggregated as a function of unidimensional indices for individual attributes and a measure of association between attributes. Intergroup or temporal variations in the overall multidimensional inequality can therefore be traced to variations in its unidimensional inequality and association measure components. The related statistical inference naturally entails joint hypothesis tests on particular subsets of those components. But the inferential framework has not been developed yet, and empirical results to date are presented without statistical tests.

This paper complements the existing literature on single hypothesis tests on unidimensional indices (e.g. Davidson and Flachaire, 2007) and the aggregated forms of multidimensional indices (Abul Naga, 2010), by developing a framework for testing joint hypotheses on unidimensional indices and an association measure that arise from decomposing a multidimensional index. Our Monte Carlo evidence suggests that, combined with bootstrapping, the proposed chi-squared tests provide reliable tools for drawing finite sample inferences.

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1. Attribute decomposable inequality indices

Consider data on p attributes of well-being in a sample of n individuals. Individual i has resources $\boldsymbol{x}_i := (x_{i1}, \ldots, x_{ip})$, where $\boldsymbol{x}_i \in \mathbb{R}^p_{++}$. We gather the data in a matrix $\boldsymbol{X} := \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix} \in \mathbb{R}^{n\times p}_{++}$, and let $\bar{\boldsymbol{x}} := \frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i = (\bar{x}_1, \ldots, \bar{x}_p)$ denote the vector of sample means.

Let $\hat{\imath} : \mathbb{R}^{n\times p}_{++} \to \mathbb{R}_+$ denote a multidimensional inequality index, $W : \mathbb{R}^{n\times p}_{++} \to \mathbb{R}$ be the social welfare function underlying the derivation of $\hat{\imath}$, and $\omega := W(\boldsymbol{X})$ be the level of welfare attained by $\boldsymbol{X}$. If W satisfies the standard axioms of anonymity, additivity across individuals, continuity, increasing monotonicity and equality preference, one may define an increasing and concave utility function $u(.)$ such that $W(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^n u(\boldsymbol{x}_i)$, and a scalar $\hat{\theta}(\boldsymbol{X})$ in the unit interval such that $u(\hat{\theta}\bar{\boldsymbol{x}}) = \frac{1}{n}\sum_{i=1}^n u(\boldsymbol{x}_i)$. The scalar $\hat{\theta}$ is an index of multidimensional *equality* in $\boldsymbol{X}$, and $\hat{\imath}(\boldsymbol{X}) := 1 - \hat{\theta}(\boldsymbol{X})$ is the corresponding inequality index.

Assume furthermore that W is scale-invariant, and consider for expositional simplicity the case of two attributes ($p = 2$) of well-being.[1] Then the utility function underlying the definition of W takes one of three possible forms (Aczel, 1988; ch. 4, Corollary 4):

$$u(x_{i1}, x_{i2}) : = x_{i1}^{\alpha} x_{i2}^{\beta} \quad \alpha, \beta > 0,\ \alpha + \beta \le 1 \tag{1}$$

$$u(x_{i1}, x_{i2}) : = -x_{i1}^{\alpha} x_{i2}^{\beta} \quad \alpha, \beta < 0 \tag{2}$$

$$u(x_{i1}, x_{i2}) : = \alpha \ln x_{i1} + \beta \ln x_{i2} \quad \alpha, \beta > 0. \tag{3}$$

A multidimensional Atkinson–Kolm–Sen (mAKS) inequality index arises from (1) and (2), while a multidimensional mean-logarithmic deviation (mMLD) inequality index arises from (3) (Tsui, 1995).

Abul Naga and Geoffard (2006) show that both mAKS and mMLD indices are decomposable by attributes. The mAKS *equality* index can be written as a function of three components, $\hat{\theta}_{mAKS}(\boldsymbol{X}) = \exp(\frac{\alpha}{(\alpha+\beta)} \ln \hat{\delta}_1 + \frac{\beta}{(\alpha+\beta)} \ln \hat{\delta}_2 + \frac{1}{(\alpha+\beta)} \ln \hat{\delta}_3)$ where

$$\hat{\delta}_1 = \left(\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha}\right)^{1/\alpha} \Big/ \bar{x}_1 \tag{4}$$

$$\hat{\delta}_2 = \left(\frac{1}{n}\sum_{i=1}^n x_{i2}^{\beta}\right)^{1/\beta} \Big/ \bar{x}_2 \tag{5}$$

$$\hat{\delta}_3 = \frac{\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha} x_{i2}^{\beta}}{\left(\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha}\right)\left(\frac{1}{n}\sum_{i=1}^n x_{i2}^{\beta}\right)}. \tag{6}$$

$\hat{\theta}_{mAKS}(\boldsymbol{X})$ is the *aggregated form* and $\hat{\boldsymbol{\delta}} = (\hat{\delta}_1\ \hat{\delta}_2\ \hat{\delta}_3)'$ the *disaggregated form* of the mAKS index. $\hat{\delta}_1$ ($\hat{\delta}_2$) is the unidimensional AKS index of *equality* in the first (second) attribute, and $\hat{\delta}_3$ is a measure of association between the two attributes.[2] The mMLD *equality* index can be written as a function of two components, $\hat{\theta}_{mMLD}(\boldsymbol{X}) = \exp(\frac{\alpha}{(\alpha+\beta)} \ln \hat{\delta}_1 + \frac{\beta}{(\alpha+\beta)} \ln \hat{\delta}_2)$, where

$$\hat{\delta}_j = \exp\left[\frac{1}{n}\sum_{i=1}^n \ln(x_{ij}) - \ln \bar{x}_j\right] \quad j = 1, 2. \tag{7}$$

$\hat{\delta}_1$ and $\hat{\delta}_2$ are unidimensional MLD *equality* indices. The mMLD index is *strongly decomposable* à la Kobus (2012), because $\hat{\theta}_{mMLD}$ is entirely characterized by the two equality indices pertaining to each attribute's marginal distribution. Since the mAKS and mMLD inequality indices are $\widehat{\imath}_{mAKS}(\boldsymbol{X}) := 1 - \hat{\theta}_{mAKS}(\boldsymbol{X})$ and $\widehat{\imath}_{mMLD}(\boldsymbol{X}) := 1 - \hat{\theta}_{mMLD}(\boldsymbol{X})$, they are also decomposable by attributes.

2. Large sample distribution

Let $\hat{\boldsymbol{\delta}}(\boldsymbol{X}) = (\hat{\delta}_1, \ldots, \hat{\delta}_k)'$ denote the vector of equality indices and measure of association in the sample.[3] For estimation and inference, it is convenient to define $\hat{\boldsymbol{\delta}}(\boldsymbol{X})$ in relation to m sample moments of $\boldsymbol{X}$,

$$\boldsymbol{s} := \begin{bmatrix} \frac{1}{n}\sum_{i=1}^n g_1(\boldsymbol{x}_i) & \cdots & \frac{1}{n}\sum_{i=1}^n g_m(\boldsymbol{x}_i) \end{bmatrix} \tag{8}$$

via a function $\boldsymbol{F} : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that $\hat{\boldsymbol{\delta}} = \boldsymbol{F}(\boldsymbol{s})$. One can then define the population indices $\boldsymbol{\delta}_o$ in relation to m population moments,

$$\boldsymbol{\sigma}_o := \begin{bmatrix} E[g_1(\boldsymbol{x}_i)] & \cdots & E[g_m(\boldsymbol{x}_i)] \end{bmatrix} \tag{9}$$

such that $\boldsymbol{\delta}_o = \boldsymbol{F}(\boldsymbol{\sigma}_o)$. For instance, in the context of the mAKS index we have

$$\boldsymbol{s} := \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha}x_{i2}^{\beta} & \frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha} & \frac{1}{n}\sum_{i=1}^n x_{i2}^{\beta} \end{bmatrix} \tag{10}$$

$$\boldsymbol{\sigma}_o := \begin{bmatrix} E(x_1) & E(x_2) & E(x_1^{\alpha}x_2^{\beta}) & E(x_1^{\alpha}) & E(x_2^{\beta}) \end{bmatrix} \tag{11}$$

and in the context of the mMLD index,

$$\boldsymbol{s} := \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \frac{1}{n}\sum_{i=1}^n \ln(x_{i1}) & \frac{1}{n}\sum_{i=1}^n \ln(x_{i2}) \end{bmatrix} \tag{12}$$

$$\boldsymbol{\sigma}_o := \begin{bmatrix} E(x_1) & E(x_2) & E(\ln x_1) & E(\ln x_2) \end{bmatrix}. \tag{13}$$

Let $\bar{g}_j$ denote the *jth* component of $\boldsymbol{s}$, i.e. $\bar{g}_j = \sum_{i=1}^n g_j(\boldsymbol{x}_i)/n$, and $\boldsymbol{Z}$ be an $n \times m$ matrix where

$$\boldsymbol{Z} := \begin{bmatrix} g_1(\boldsymbol{x}_1) - \bar{g}_1 & \cdots & g_m(\boldsymbol{x}_1) - \bar{g}_m \\ \vdots & & \vdots \\ g_1(\boldsymbol{x}_n) - \bar{g}_1 & \cdots & g_m(\boldsymbol{x}_n) - \bar{g}_m \end{bmatrix}. \tag{14}$$

Define also the $k \times m$ Jacobian matrix $\boldsymbol{J} := \partial \boldsymbol{F}/\partial \boldsymbol{\sigma}$,

$$\boldsymbol{J} := \begin{pmatrix} \partial \boldsymbol{F}_1/\partial\sigma_1 & \cdots & \partial \boldsymbol{F}_1/\partial\sigma_m \\ & \vdots & \\ \partial \boldsymbol{F}_k/\partial\sigma_1 & \cdots & \partial \boldsymbol{F}_k/\partial\sigma_m \end{pmatrix}. \tag{15}$$

where σ_j refers to the *jth* component of $\boldsymbol{\sigma}_o$.

Under appropriate assumptions, the Continuous Mapping Theorem ensures that $\hat{\boldsymbol{\delta}} = \boldsymbol{F}(\boldsymbol{s})$ is a consistent estimator of $\boldsymbol{\delta}_o = \boldsymbol{F}(\boldsymbol{\sigma}_o)$. Specifically, consider the following assumptions:

[1] Let $\boldsymbol{Y} \in \mathbb{R}^{n\times p}_{++}$ be another data matrix and Λ be a $p \times p$ positive-definite diagonal matrix. $W(.)$ is scale-invariant when $W(\boldsymbol{X}) = W(\boldsymbol{Y})$ if and only if $W(\boldsymbol{X\Lambda}) = W(\boldsymbol{Y\Lambda})$.

[2] As in the multidimensional case, $1 - \widehat{\delta}_1$ and $1 - \widehat{\delta}_2$ are the corresponding inequality indices.

[3] In general, $k = p + 1$ as the decomposition produces p equality indices together with an association measure. However, $k = p$ when the multidimensional index is strongly decomposable (as is the case in the context of $\hat{\theta}_{mMLD}$).