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Multidimensional endogenous gridpoint method: Solving triangular dynamic stochastic optimization problems without root-finding operations

Fedor Iskhakov

Center of Excellence of Population Aging Research (CEPAR), University of New South Wales, 223 Anzac Parade, Kensington NSW 2033, Australia

HIGHLIGHTS

- Certain dynamic stochastic models can be solved without root fining operations.
- The paper describes a class of such models using five conditions.
- The solution method is multidimensional endogenous gridpoint method (EGM).
- Typical member of this class is a model of multiple stock dynamics.

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ABSTRACT

This paper defines the class of triangular dynamic stochastic optimization problems with multiple continuous choices which can be solved by the multidimensional generalization of the method of endogenous gridpoints without costly root-finding operations. The typical member of this class is a model of multiple asset dynamics, with potential applications in wealth, health and human capital accumulation, portfolio problems, multisector growth models, etc.

1. Introduction

The endogenous gridpoint method (EGM) introduced by Carroll (2006) to solve stochastic dynamic models with one continuous state variable and a single continuous control, proved to be very fast and accurate[1]—mainly due to its ability to solve the first order conditions directly without root-finding operations. The method had been generalized to a number of settings,[2] where it proved to retain its numerical advantage over the traditional dynamic programming approach. Yet, the applicability of EGM method had been shown only for a collection of particular economic models, whereas the class of models which can be solved without root-finding operations has not been rigorously characterized.[3]

This paper proposes five conditions which are jointly sufficient for a given problem to belong to the class of *triangular* dynamic optimization problems that can be solved by the multidimensional version of EGM in such a way that all root-finding operations are

E-mail address: f.iskhakov@unsw.edu.au.

[1] As a result, the computation of the solution for a classic consumption–savings model that could "easily done on a 386-series PC, taking 5–20 min per calculation" by Deaton (1991, p. 1229), takes only a fraction of a second on a modern laptop (Jørgensen, 2013, p. 289), with the speedup of approximately two orders of magnitude from the use of EGM when compared to traditional value functions iterations. Barillas and Fernandez-Villaverde (2007) and Iskhakov et al. (2015) find similar magnitude of the speedup with no effect on the solution accuracy.

[2] See Barillas and Fernandez-Villaverde (2007), Hintermaier and Koeniger (2010), Ludwig and Schön (2014), Fella (2014) and Iskhakov et al. (2015) for various applications of EGM framework.

[3] White (2015) develops a general theory of EGM, which does not however aim at defining the class of problems solvable without root-finding operations.

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avoided. The key insight is to ensure that the system of Euler equations that characterizes the solution has a triangular structure, and thus can be solved directly using forward substitution under standard EGM assumptions of analytical invertibility of marginal utilities.

To simplify the exposition, in this paper I only consider unconstrained choice problems. The standard approach of treating corner solutions which relies on the monotonicity of the policy function[4] or Inada (1963) derivative conditions can be generalized with additional assumptions, but general treatment based on Kuhn–Tucker conditions used by Hintermaier and Koeniger (2010) is also available for particular applications. It should also be noted that the proposed multidimensional EGM does not break the curse of dimensionality—multidimensional integration and interpolation on irregular grids are inevitable in these models.[5]

2. Two-dimensional illustrative example

Consider a discrete time[6] dynamic optimization problem with two state variables X_t and Y_t, and two continuous decision variables x_t and y_t, which represent, for example, the choice of consumption and the exercise intensity in the model of health and wealth dynamics. Denoting the instantaneous utility function $u(x_t, y_t)$ and the discount factor β, the Bellman equation of the problem is

$$V_t(X_t, Y_t) = \max_{x_t, y_t} \{u(x_t, y_t) + \beta E [V_{t+1}(X_{t+1}, Y_{t+1}) | X_t, Y_t, x_t, y_t]\}, \quad t \in \{1, \dots, T-1\}. \tag{1}$$

Assume that the transition rules for the states X_t and Y_t can be written as

$$\begin{aligned} X_{t+1} &= \chi\left(f(X_t, x_t), g(Y_t, y_t), \xi_{t+1}\right), \\ Y_{t+1} &= \psi\left(f(X_t, x_t), g(Y_t, y_t), \xi_{t+1}\right), \end{aligned} \tag{2}$$

in other words *sufficient statistics* $f(t) = f(X_t, x_t)$ and $g(t) = g(Y_t, y_t)$ contain all the information about period t that determines the distribution of values of the states in period $t+1$. Following the tradition in operations research (Powell, 2007, pp. 129–144), I refer to $\big(f(t), g(t)\big)$ as *post-decision states*. For simplicity, assume that functions χ, ψ, f and g are time invariant.[7] The expectation in (1) is taken over the joint distribution of idiosyncratic shocks $\xi_{t+1} \in \mathbb{R}^K$, and because of (2) can be written conditional on $\big(f(t), g(t)\big)$.

The solution of the problem given by the family of policy functions $\delta_t : (X_t, Y_t) \to (x_t, y_t)$, $t \in \{1, \dots, T-1\}$ satisfies the system of first order conditions for (1)

$$\begin{cases} u'_x(x_t, y_t) + \beta f'_x(t) \\ \quad \times E\left[\dfrac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_f + \dfrac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_f \middle| f(t), g(t) \right] = 0, \\ u'_y(x_t, y_t) + \beta g'_y(t) \\ \quad \times E\left[\dfrac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_g + \dfrac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_g \middle| f(t), g(t) \right] = 0. \end{cases} \tag{3}$$

By envelope theorem we have

$$\begin{aligned} \frac{\partial V_t}{\partial X_t} &= \beta f'_X(t) E\left[\frac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_f + \frac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_f \middle| f(t), g(t) \right] \\ &= -\frac{f'_X(t)}{f'_x(t)} u'_x(x_t, y_t), \\ \frac{\partial V_t}{\partial Y_t} &= \beta g'_Y(t) E\left[\frac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_g + \frac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_g \middle| f(t), g(t) \right] \\ &= -\frac{g'_Y(t)}{g'_y(t)} u'_y(x_t, y_t). \end{aligned} \tag{4}$$

Combining (4) with (3), we derive the system of Euler equations

$$\begin{cases} u'_x(x_t, y_t) = \beta f'_x(t) E\left[\chi'_f \dfrac{f'_X(t+1)}{f'_x(t+1)} u'_x(x_{t+1}, y_{t+1}) \right. \\ \qquad \left. + \psi'_f \dfrac{g'_Y(t+1)}{g'_y(t+1)} u'_y(x_{t+1}, y_{t+1}) \middle| f(t), g(t) \right], \\ u'_y(x_t, y_t) = \beta g'_y(t) E\left[\chi'_g \dfrac{f'_X(t+1)}{f'_x(t+1)} u'_x(x_{t+1}, y_{t+1}) \right. \\ \qquad \left. + \psi'_g \dfrac{g'_Y(t+1)}{g'_y(t+1)} u'_y(x_{t+1}, y_{t+1}) \middle| f(t), g(t) \right], \end{cases} \tag{5}$$

which constitutes the set of necessary conditions linking the optimal choices (x_t, y_t) in period t to the optimal choices (x_{t+1}, y_{t+1}) in $t+1$, given $\big(f(t), g(t)\big)$. Note that (x_t, y_t) does not appear in the right hand side (RHS) in (5).

Given an appropriate terminal condition, for example $\delta_T(X_T, Y_T) = (X_T, 0)$ in the hypothetical model of health and wealth dynamics, the EGM method is implemented by fixing the grid over $\big(f(t), g(t)\big)$; computing RHS in (5) for each point of this grid, using the next period policy function δ_{t+1} to find (x_{t+1}, y_{t+1}); backing out the optimal decision by solving (5) for (x_t, y_t); and finally recovering the endogenous state point (X_t, Y_t) from post-decision states and optimal decision. Repeating the latter steps for all points in the grid over $\big(f(t), g(t)\big)$ yields an approximation of the policy function δ_t, and the backward induction continues to the period $t-1$.

The numerical efficiency of EGM in two-dimensional problems satisfying (2) hinges on whether the system (5) can be solved for (x_t, y_t) by direct computation. The original one-dimensional EGM relies on the analytical invertibility of the marginal utility function, which is enough to yield the optimal period t choice by direct computation when (5) only has one equation. Clearly, the invertibility of all the partial derivatives is insufficient in the multidimensional case. Because (5) is a system of non-linear equations, even the existence of the solution is not guaranteed without additional conditions on the utility function.

3. *M*-dimensional triangular dynamic optimization problems

Consider now an optimization problem with M continuous choice variables $x_t = (x_t^1, \dots, x_t^M) \in \mathbb{R}^M$ which govern the transitions of M continuous state variables $X_t = (X_t^1, \dots, X_t^M) \in \mathbb{R}^M$. Let $s_t \in \mathbb{R}^N$ denote additional state variables which follow *uncontrolled* N-dimensional Markov processes, and therefore will be referred to as exogenous states.[8] The following assumptions are needed to derive the main result below.

[4] Used by Carroll (2006), Fella (2014) and Iskhakov et al. (2015).

[5] Ludwig and Schön (2014) show that already in two-dimensional case the computational cost of Delaunay triangulation may offset the speed advantage of the EGM approach. However, White (2015) proposes an interpolation method which takes into account the ordinal information about the endogenous gridpoints produced by EGM. Orthogonal polynomial approximation (Judd, 1998, pp. 202–223) may be suggested as another useful interpolation method.

[6] Both infinite and finite horizon cases are included assuming that the infinite horizon model is solved by time iterations indexed with t, and $T = +\infty$.

[7] It is straightforward to extend the analysis to time varying specifications.

[8] To simplify the exposition, they will be treated as discrete. In applications these states are either discrete or discretized as in Tauchen (1986).