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On the identification of multivariate correlated unobserved components models[☆]

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ABSTRACT

This letter analyses identification for multivariate unobserved components models with correlated trend and cycle innovations. We address both the order as well as the rank criteria. Identification is shown for lag structures with lengths larger than one. We also discuss UC models with common features and with cycles that allow for dynamic spillovers.

1. Introduction

Traditionally, in unobserved components (UC) models with stochastic trend and autoregressive (AR) cycle the innovations to the state variables were assumed to be uncorrelated, e.g. Harvey (1985) and Clark (1987). Balke and Wohar (2002) and Morley et al. (2003) allowed for correlation of the UC shocks. The latter authors state that identification in a univariate setting can be achieved if the lag polynomial in the cycle is at least of second order.

This letter demonstrates how identification can be derived for multivariate correlated UC models. We clarify the role of identification of the reduced-form vector autoregressive integrated moving average (VARIMA) model and present a rigorous treatment of the order and rank conditions. To the best of our knowledge identification of correlated UC models has only been discussed with respect to the order condition with the exception of the univariate UC model in Morley et al. (2003).

We find identification for given lag lengths larger than one. This criterion has to be fulfilled for all cyclical components, but the lag structure does not have to be complete. Furthermore, we briefly address UC models with common features and extend the usual UC specification to the case of a non-diagonal VAR cycle. Our results can provide a useful basis for growing strands of literature that apply and develop correlated UC models, e.g. Morley (2007), Sinclair (2009), Startz and Tsang (2010), Weber (2011) and Klinger and Weber (2014).

2. Correlated unobserved component model

Consider the following correlated UC model of for the $K \times 1$ random vector y_t, see Morley et al. (2003) and Sinclair (2009) for univariate and multivariate cases, respectively,

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \mu + \tau_{t-1} + \eta_t \\ c_t &= B_1 c_{t-1} + \cdots + B_p c_{t-p} + \varepsilon_t, \end{aligned} \tag{1}$$

with

$$v_t = \begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim \text{i.i.d. } N(0, \Sigma_v), \tag{2}$$

where

$$\Sigma_v = \begin{pmatrix} \Sigma_\eta & \Sigma_{\eta\varepsilon} \\ \Sigma'_{\eta\varepsilon} & \Sigma_\varepsilon \end{pmatrix}.$$

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Thus, the trend component τ_t follows a multivariate random walk, while the cyclical component c_t has a vector autoregressive (VAR) structure for which we make the following assumptions. The parameter matrices $B_1, \dots, B_p$ are diagonal with typical diagonal elements $b_{kk,i},\ i = 1, \dots, p, k = 1, \dots, K$, and $B_p \neq 0$ such that

$$|I_K - B_1 z - \cdots - B_p z^p| \neq 0 \quad \text{for } |z| \leq 1. \tag{3}$$

Consequently, the cyclical part of each component in y_t is characterized by a stable AR process of order at most p. This UC model framework is labelled UC-VAR(p) in the following. We will address the case of a general non-diagonal VAR(p) cycle in Section 3.5.

The set-up (1)–(3) results in a reduced-form VARIMA(p, 1, p) representation. Its canonical form, compare e.g. Schleicher (2003), reads as

$$\begin{aligned} B(L)\Delta y_t &= B(1)\mu + B(L)\eta_t + \Delta\varepsilon_t & (4)\\ &= c + \Theta(L)u_t, & (5) \end{aligned}$$

where $B(L) = B_0 - B_1 L - \cdots - B_p L^p$ and $\Theta(L) = \Theta_0 + \Theta_1 L + \cdots + \Theta_p L^p$ are K-dimensional lag-polynomials of order p with $B_0 = \Theta_0 = I_K$, and $c = B(1)\mu$. The polynomials in row i and column j of $B(L)$ and $\Theta(L)$ will be denoted by $b_{ij}(L)$ and $\theta_{ij}(L)$, respectively. Accordingly, the i-th row of $\Theta(L)$ is given by $\Theta_{i\bullet}(L) = [\theta_{i1}(L), \dots, \theta_{iK}(L)]$. Moreover, we have $u_t \sim$ i.i.d. $N(0, \Sigma_u)$. The representation (5) is due to a multivariate version of Granger's Lemma, compare e.g. Lütkepohl (1984, Lemma 1). The reduced-form autocovariance structure of the vector MA part $m_t = \Theta(L)u_t$ is described by the matrices $\Gamma_h = \mathrm{E}(m_t m_{t-h}') = \sum_{i=0}^{p-h} \Theta_{i+h}\Sigma_u\Theta_i'$, $h = 1, 2, \dots$, such that $\Gamma_h = 0$ for $h > p$.

3. Identification of multivariate UC models

3.1. Identification approach

We analyse whether the parameters of a given structural UC model with a diagonal cycle can be identified from its implied reduced-form VARIMA (5). The label 'given UC model' refers to a given set of orders $\{p_1, \dots, p_K\}$ of the individual AR cycles. This is a common assumption in the literature, see e.g. Hotta (1989), Morley et al. (2003) and Sinclair (2009).

Moreover, we assume that the lag orders of the individual AR cycles in the UC-VAR(p) model are minimal in the sense that y_t has no reduced form VARIMA representation with lag orders $p_k^* < p_k$ for at least one $k = 1, \dots, K$. Then, there are no common roots to $b_{kk}(z)$ and $\Theta_{k\bullet}(z)$, $k = 1, \dots, K$, i.e. there exists no value z^* such that $b_{kk}(z^*) = 0$ and $\Theta_{k\bullet}(z^*) = 0$. As a consequence, the reduced form VARIMA in (5) is identified. This follows from Dufour and Pelletier (2011, Assumption 3.13, Theorem 3.14) as the VAR component is diagonal and $B_0 = \Theta_0 = I_K$.

Hence, the VAR component in (5) can be uniquely separated from the MA part such that the reduced-form VAR parameters can also be regarded as given.[2] This allows us to discuss identification within a system of linear equations that relates reduced-form and structural variance parameters by dealing with the necessary order and sufficient rank conditions in the following.

3.2. Order condition

The UC model (1)–(3) satisfies the order condition for identification as can be seen as follows. The Kp parameters in $B_1, \dots, B_p$ are always identified since the VAR polynomial can be obtained from the reduced form (5). Therefore, the parameter vector μ is also identified. There remain $2K^2 + K$ structural variance parameters in Σ_v to be identified. Equating the autocovariance structures of $B(L)\eta_t + \Delta\varepsilon_t$ and $\Theta(L)u_t$ provides us with a link of these structural parameters to the reduced-form variance parameters, compare (4) and (5). Due to the symmetry of Γ_0, the reduced form contains $(K^2 + K)/2 + K^2 p$ pieces of variance information in $\Gamma_0, \Gamma_1, \dots, \Gamma_p$. Thus, the order condition is satisfied for $p \geq 2$ since $[(K^2 + K)/2 + K^2 p] \geq [2K^2 + K]$ in this case. Except for $p = 2$ and $K = 1$, there are more reduced-form than structural-form variance parameters.

3.3. Rank condition: diagonal VAR(2) cycle

It remains to show that the structural variance parameters in Σ_v can indeed be uniquely recovered from the reduced-form VMA variance matrices $\Gamma_0, \Gamma_1, \dots, \Gamma_p$. This requires to meet the relevant rank condition related to the system of equations linking the reduced-form and structural parameters. To simplify the understanding of the corresponding proof we start with the case $p = 2$. The proof relies on the construction of a particular submatrix that has full rank. This is easily achieved for a diagonal VAR(2) by assumption.

From the equivalence of (4) and (5) we first obtain

$$\begin{aligned}
\mathrm{vec}(\Gamma_0) &= \gamma_0 = [I_{K^2} + (B_1 \otimes B_1) \\
&\quad + (B_2 \otimes B_2)]\mathrm{vec}(\Sigma_\eta) + 2\mathrm{vec}(\Sigma_\varepsilon) \\
&\quad + \left[(I_{K^2} + C_{KK}) + (I_K \otimes B_1) \right.\\
&\quad \left. + C_{KK}(I_K \otimes B_1)\right]\mathrm{vec}(\Sigma_{\eta\varepsilon}) \\
\mathrm{vec}(\Gamma_1) &= \gamma_1 = [-(I_K \otimes B_1) \\
&\quad + (B_1 \otimes B_2)]\mathrm{vec}(\Sigma_\eta) - \mathrm{vec}(\Sigma_\varepsilon) \\
&\quad - [C_{KK} + (I_K \otimes B_1) - (I_K \otimes B_2)]\mathrm{vec}(\Sigma_{\eta\varepsilon}) \\
\mathrm{vec}(\Gamma_2) &= \gamma_2 = -(I_K \otimes B_2)\mathrm{vec}(\Sigma_\eta) - (I_K \otimes B_2)\mathrm{vec}(\Sigma_{\eta\varepsilon}),
\end{aligned} \tag{6}$$

where the vec-operator stacks the columns of a matrix below each other and C_{mn} is the $(mn \times mn)$ commutation matrix with $\mathrm{vec}(A') = C_{mn}\mathrm{vec}(A)$ for any $(m \times n)$ matrix A.

As Γ_0 is symmetric, γ_0 can just provide $\frac{1}{2}K(K+1)$ linearly independent equations. Therefore, we consider $\gamma_0^* = \mathrm{vech}(\Gamma_0)$ in the following, where the vech-operator is defined to stack columnwise the elements on and below the main diagonal of a square matrix below each other. Let D_K be the $(K^2 \times \frac{1}{2}K(K+1))$ duplication matrix such that $\mathrm{vec}(A) = D_K\mathrm{vech}(A)$ for any symmetric $(K \times K)$ matrix A and define $D_K^+ = (D_K' D_K)^{-1} D_K'$. Since $D_K^+\mathrm{vec}(A) = \mathrm{vech}(A)$ if A is symmetric, see Lütkepohl (1996, Section 9.5), we can re-write system (6) as

$$\gamma^* = B^*\sigma^*, \tag{7}$$

where $\gamma^* = [\gamma_0^{*\prime} : \gamma_1' : \gamma_2']'$, $\gamma_0^* = D_K^+\gamma_0$, $\sigma^* = [\mathrm{vech}(\Sigma_\eta)' : \mathrm{vec}(\Sigma_\varepsilon)' : \mathrm{vec}(\Sigma_{\eta\varepsilon})']'$, and

$$B^* = \begin{bmatrix} D_K^+(I_{K^2} + B_1 \otimes B_1 + B_2 \otimes B_2)D_K & 2D_K^+ & 2D_K^+(I_{K^2} + I_K \otimes B_1) \\ (-I_K \otimes B_1 + B_1 \otimes B_2)D_K & -I_{K^2} & -(C_{KK} + I_K \otimes B_1 - I_K \otimes B_2) \\ -(I_K \otimes B_2)D_K & 0_{(K^2 \times K^2)} & -I_K \otimes B_2 \end{bmatrix}. \tag{8}$$

We will show below that the $(K^* \times K^*)$ matrix B^*, $K^* = 2.5K^2 + 0.5K$, has full rank. Hence, the structural variance parameters can be recovered from the reduced-form parameters by $\sigma^* = B^{*-1}\gamma^*$. Note that the off-diagonal elements of the symmetric variance matrix Σ_ε appear twice in σ^*. However, we analyse identification with respect to a given UC model. Thus, γ^* contains the implied, i.e. correct, reduced-form VMA variance parameters such that $B^{*-1}\gamma^*$ indeed returns two identical sets of off-diagonal elements of Σ_ε.

[2] Further details on the importance of identifying the reduced form VARIMA representation in our context and on the relationship to VARMA cycles can be found in Trenkler and Weber (2015).