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On the existence of stable population in life cycle models

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HIGHLIGHTS

- We study the demography of life cycle general equilibrium models.
- The existence of stable population is established.
- This result is crucial for the aggregation of individual decisions in this class of models.

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ABSTRACT

A common assumption adopted in life cycle general equilibrium models is that the population is stable at steady state, that is, its relative age distribution becomes constant over time. An open question is whether the demographic assumptions commonly adopted in these models in fact imply that the population becomes stable. In this article we prove the existence of a stable population in a demographic environment where both the age-specific mortality rates and the population growth rate are constant over time, the setup commonly adopted in life cycle general equilibrium models. Hence, the stability of the population do not need to be taken as assumption in these models.

1. Introduction

Quantitative life cycle general equilibrium models are one of the main tools used by economists and policy makers to conduct economic analysis and *ex ante* policy evaluations. The use of these models started with the seminal works of Imrohoroğlu et al. (1995) and Huggett (1996). A common assumption adopted in these models is that the population is stable at steady state, that is, its relative age distribution becomes constant over time. This assumption is important because it allows the calculation of a stationary distribution of individuals, which in turn permits the aggregation of individual decisions. An open question is whether we can prove the existence of a stable population from the demographic assumptions commonly adopted in these models.

A natural approach to prove this result would be to use the *Ergodic Theorems of Demography*, which is a set of results that provides conditions for the stability of a population.[1] These theorems make assumptions about the age-specific fertility and mortality rates of a population. However, these results cannot be applied to life cycle general equilibrium models, as these models are silent about fertility rates. These rates do not even appear in this type of model. In contrast, life cycle models make assumptions about the rate of population growth, assuming that it is constant over time, while the theorems are silent about this rate.

In this article we prove the existence of a stable population in a demographic environment where both the age-specific mortality rates and the population growth rate are constant over time, the setup commonly adopted in life cycle general equilibrium models. We make no assumptions about fertility rates. To our knowledge, no formal proof of this result exists in the economics

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[1] For references on stable population theory, see Sykes (1969), McFarland (1969), Parlett (1970), Cohen (1979), Arthur (1981), and Arthur (1982).

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and demographics literature. To do this, we model a population in a generic transition path, that is, out of the steady state, and show that the population becomes stable when reaching the steady state. Important results of the theories of difference equations and polynomials are used in the formalization. We then simulate computationally the dynamics of a population to illustrate this theoretical result.

2. Demographic environment

Time is discrete and denoted by $t \in \{0, 1, 2, \dots\}$. The population grows at a constant rate over time, where the growth factor is denoted by $g > 1$.[2] The age of individuals is denoted by $j \in \{0, \dots, J\}$. The survival probability from age j to age $(j+1)$ is given by p_j, which is constant over time. Individuals younger than J have strictly positive survival probability, that is, $p_j \in (0, 1]$ for all $j < J$.[3] Individuals die with certainty at the end of age J, which means that $p_J = 0$. The unconditional probability of being alive at age $j > 0$ is given by $q_j = p_0 \dots p_{j-1}$, where $q_0 = 1$. The number of individuals with age j at time t is denoted by $N_{j,t}$. The size of the population at time t is given by $N_t = N_{0,t} + \dots + N_{J,t}$. The share of individuals with age j at time t is given by $M_{j,t} = N_{j,t}/N_t$. Because only a fraction p_j of the population with age j survives until age $(j+1)$, we have that $N_{j+1,t+1} = p_j N_{j,t}$ for all (j, t). Because the population grows at a constant factor g each period of time, we have that $N_{t+1} = gN_t$ for all t. The proposition below states two important results regarding the dynamics of the population.

Proposition 1. *Given the demographic environment, the following results are true:*

(i) $M_{j+1,t+1} = (p_j/g)M_{j,t}$ *for all* (j, t);
(ii) $M_{j,t} = (q_j/g^j)M_{0,t-j}$ *for all* (j, t) *such that* $j \le t$.

Proof. To prove the first result, note that because $N_{j+1,t+1} = p_j N_{j,t}$ for all (j, t) and $N_{t+1} = gN_t$ for all t, we have that

$$M_{j+1,t+1} = \frac{N_{j+1,t+1}}{N_{t+1}} = \frac{p_j N_{j,t}}{gN_t} = \frac{p_j}{g} M_{j,t}.$$

To prove the second result, we must apply the first result recursively and use the definition of q_j. By doing this, we conclude that for all (j, t) such that $j \le t$ it is true that

$$\begin{aligned} M_{j,t} &= \frac{p_{j-1}}{g} M_{j-1,t-1} = \frac{p_{j-2}p_{j-1}}{g^2} M_{j-2,t-2} \\ &= \dots = \frac{p_0 \cdots p_{j-1}}{g^j} M_{0,t-j} = \frac{q_j}{g^j} M_{0,t-j}. \quad \square \end{aligned}$$

3. Stability of population

In this section we show that our demographic environment implies the existence of a stable population in the steady state. We start by defining the concept of stable population.

Definition 1. A population is called stable if its relative age distribution is constant over time.

By the definition above, we must show that the sequences $\{M_{j,t}\}$ are convergent for all j. The proposition below provides a sufficient condition for the convergence of all these sequences.

Proposition 2. *If the sequence* $\{M_{0,t}\}$ *is convergent, then the sequences* $\{M_{j,t}\}$ *are convergent for all* $j > 0$.

Proof. The result follows from the item (ii) of Proposition 1. $\square$

The above result ensures that if the share of newborns converges, then the share of all other ages also converges, which implies that the relative age distribution of the whole population is convergent. Therefore, our objective now is to prove that the share of newborns converges, i.e., that the sequence $\{M_{0,t}\}$ is indeed convergent. The first step is to show that we can express $M_{0,t}$ as a difference equation for all t. Using the definition of N_t, we can write that

$$N_{0,t} + N_{1,t} + N_{2,t} + \dots + N_{J,t} = N_t.$$

Dividing both sides of the above expression by N_t and using the item (ii) of Proposition 1, we conclude that

$$M_{0,t} + \frac{q_1}{g} M_{0,t-1} + \frac{q_2}{g^2} M_{0,t-2} + \dots + \frac{q_J}{g^J} M_{0,t-J} = 1$$

for all t such that $J \le t$. For ease of notation, define $c_j = q_j/g^j$ for all j. Thus, we can write the above difference equation as

$$c_0 M_{0,t} + c_1 M_{0,t-1} + c_2 M_{0,t-2} + \dots + c_J M_{0,t-J} = 1. \tag{1}$$

From the theory of difference equations, we know that the general solution of Eq. (1) is given by $M_{0,t} = P_t + H_t$, where P_t is called the particular solution and H_t is called the homogeneous solution. To find the particular solution P_t we use the *guess and verify* method. First, we assume that the particular solution is a constant P that does not depend on t. Then, to check that this is the case, we replace $M_{0,t-j}$ by P in equation (1) and solve for P. By doing this, we find that

$$P_t = P = \frac{1}{c_0 + c_1 + c_2 + \dots + c_J}. \tag{2}$$

To find the homogeneous solution H_t, we need to work with the homogeneous equation associated with equation (1), which is given by

$$c_0 M_{0,t} + c_1 M_{0,t-1} + c_2 M_{0,t-2} + \dots + c_J M_{0,t-J} = 0. \tag{3}$$

The characteristic polynomial associated with the homogeneous equation (3) is given by

$$c_0 \lambda^J + c_1 \lambda^{J-1} + c_2 \lambda^{J-2} + \dots + c_J = 0, \tag{4}$$

where λ is a complex number. Let $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ be the distinct roots of Eq. (4), with multiplicities given by $\{m_1, m_2, \dots, m_r\}$. From Corollary 2.24 of Elaydi (2005), we have that the general solution of the homogeneous equation (3) is given by

$$H_t = \sum_{i=1}^{r} \lambda_i^t \left(a_{i,0} + a_{i,1} t + a_{i,2} t^2 + \dots + a_{i,m_i-1} t^{m_i-1} \right), \tag{5}$$

where $\{a_{i,0}, a_{i,1}, \dots, a_{i,m_i-1}\}$ are complex numbers for all i.

Using the results (2) and (5), we can write the general solution of Eq. (1) as

$$\begin{aligned} M_{0,t} = &\frac{1}{c_0 + c_1 + c_2 + \dots + c_J} \\ &+ \sum_{i=1}^{r} \lambda_i^t \left(a_{i,0} + a_{i,1} t + a_{i,2} t^2 + \dots + a_{i,m_i-1} t^{m_i-1} \right). \end{aligned} \tag{6}$$

Note that the above general expression for the share of newborns depends on the population growth rate, survival probabilities, and initial population. It depends on the population growth rate and survival probabilities through the constants $\{c_0, \dots, c_J\}$ and roots

[2] Our main result remains true if we consider a constant growth rate of newborns rather than a constant population growth rate.

[3] The assumption that $g > 1$ means that the population growth rate is strictly positive. We could relax this assumption and assume that $g \ge 1$ to consider zero population growth. In this case, to prove the result, we would have to assume that $p_0 < 1$ or $p_{J-1} < 1$.