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# The existence and efficiency of general equilibrium with incomplete markets under Knightian uncertainty

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## HIGHLIGHTS

- We study general equilibrium theory of incomplete markets under Knightian uncertainty.
- The existence of equilibrium is established.
- The equilibrium is shown to be constrained Pareto efficient.

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## ABSTRACT

This paper first establishes the existence of equilibrium for an economy with Knightian uncertainty and incomplete markets, and then demonstrates the constrained Pareto efficiency of the equilibrium when there is one commodity only in each state of nature.



## 1. Introduction

As is well known, Knight (1921) argues that a distinction should be made between risk and uncertainty, and claims that it is uncertainty that prevails in the real world. One way of expressing this distinction is formalized in Bewley (2002) by assuming the existence of a set of probability measures on a state space such that one state-contingent consumption bundle is preferred to another if and only if it has larger expected utility for every measure in that set. According to this formalization, the notion of risk refers to the situation when the set is a singleton, and uncertainty otherwise. Such description of uncertainty is now usually called Knightian uncertainty. It is intuitively obvious that this uncertainty will have an effect on the behavior of economic agents, and it is because of this intuition that Rigotti and Shannon (2005) make a study of the general equilibrium theory of complete markets under Knightian uncertainty. The purpose of this paper is to continue their analysis by allowing for incomplete markets.

There are several reasons for the market being incomplete, as for instance asymmetric information, moral hazard, and transaction costs (Geanakoplos, 1990). The general equilibrium theory of incomplete markets with standard expected utility (or more generally, with complete preferences) has been extensively studied (Magill and Quinzii, 2002), and a number of interesting results concerning existence of equilibrium and its efficiency have been obtained: for example, Hart (1975) shows that an equilibrium may not exist under standard preference setup, and even when it exists, may not be Pareto efficient. With this background it appears natural to examine the general equilibrium theory when both Knightian uncertainty and incomplete markets present themselves. The plan of this paper is as follows. Section 2 describes the model, Section 3 establishes the existence of equilibrium, and Section 4 studies its efficiency. Concerning the latter, a general equilibrium with incomplete markets, as said above, is generally

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speaking not Pareto efficient; but when there is one commodity only in each state of nature, it will be shown in Section 4 to be constrained Pareto efficient. In this section we use, following Magill and Quinzii (2002, Chapter 2, pp. 108–113), a functional-theoretic formalism, which, albeit somewhat abstract, makes the results and their proofs assume a very elegant form. In a sense it is this elegance that justifies the study of the one-commodity model.

## 2. The model

We study a pure exchange economy with two dates, denoted 0 and 1, and $S$ possible states of nature at date 1. We index the states by $s$ running from 1 to $S$, and for notational convenience, call date 0 state 0. Let there be $m$ consumers, $J$ assets with $J < S$, and $L$ commodities in each state. Suppose that every consumer has $X = R_+^{(S+1)L}$ as his consumption space and that consumer $i$ has $\omega^i \in X$ as his endowment vector. For $x \in X$, it is often convenient to write it state-wise as $x = (x_0, \ldots, x_S)$ with $x_s \in R^L$.

For preferences, let $\succ_i$ be the preference for consumer $i$, and $\Delta_S$ the set of probability measures on $\{1, \ldots, S\}$. We assume that there exists for every consumer $i$ a closed, convex subset $\Pi^i$ of $\Delta_S$ such that

$$x^i \succ_i (x^i)' \quad \text{if, and only if, } U_\pi^i(x^i) > U_\pi^i((x^i)') \text{ for all } \pi \in \Pi^i,$$

where $U_\pi^i(x^i) = u^i(x_0^i) + \sum_{s=1}^S \pi_s u^i(x_s^i)$ and $u^i$ is a real-valued, strictly increasing, and concave function on $R_+^L$. For a behavioral foundation of this representation see Bewley (2002). According to Rigotti and Shannon (2005, p. 237), every $\succ_i$ has an open graph; in what follows we let $P^i(x^i) = \{(x^i)' \in X \mid (x^i)' \succ_i x^i\}$.

We now discuss payoffs of the assets and budget sets of the consumers. Let $q \in R^J$ denote a price vector of the assets and $p = (p_0, \ldots, p_S) \in R_+^{(S+1)L}$ that of the commodities with $p_s$ the spot price vector in state $s$. Let $A^j = (A_1^j, \ldots, A_S^j) \in R_+^{SL}$ be the payoff vector of asset $j$ where $A_s^j \in R_+^L$ denotes its promise of the $L$ commodities in state $s$. It is natural to require that for every asset $j$ there exists an $s$ such that $A_s^j \neq 0$. The nominal return matrix at the price vector $p$ of the assets is then given by

$$V(p) = \begin{bmatrix} p_1 A_1^1 & \cdots & p_1 A_1^J \\ \vdots & & \vdots \\ p_S A_S^1 & \cdots & p_S A_S^J \end{bmatrix} = \begin{bmatrix} V_1(p_1) \\ \vdots \\ V_S(p_S) \end{bmatrix}.$$

Let $\theta^i \in R^J$ be consumer $i$'s portfolio, which gives the number of units of each of the $J$ assets he purchases. Then his budget set $B^i(p, q)$ is defined to consist of all $(x^i, \theta^i)$ satisfying

$$p_0(x_0^i - \omega_0^i) + q\theta^i \le 0$$
$$p_s(x_s^i - \omega_s^i) - V_s(p_s)\theta^i \le 0, \quad s = 1, \ldots, S.$$

With these preparations we can define the notion of general equilibrium with incomplete markets, henceforth called GEI equilibrium.

**Definition 1.** A GEI equilibrium is a list $((x^i, \theta^i)_{i=1}^m, p, q)$ satisfying

(I) there exists no $((x^i)', (\theta^i)') \in B^i(p, q)$ such that $(x^i)' \succ_i x^i$ for $i = 1, \ldots, m$;
(II) $\sum_i (x^i - \omega^i) = 0$;
(III) $\sum_i \theta^i = 0$.

## 3. Existence of equilibrium

It has been shown in Hart (1975) that equilibria may not exist without any restraint on the allowable volume of trade in assets. For this reason and following Radner (1972), we simply place the constraint $\|\theta^i\|_\infty < \tau$ for every $i$, where $\tau$ is a positive scalar and $\|\cdot\|_\infty$ denotes the $\infty$-norm. Under this condition we can deduce that

**Proposition 1.** *The model of Section 2 has a GEI equilibrium.*

**Proof.** The proof follows the line of argument of Gale and Mas-Colell (1975), and therefore we shall give a brief sketch only and leave the details to the interested reader.

We begin with the introduction of some notation. Let $x^0 = \sum_{i=1}^m x^i$, $\omega^0 = \sum_{i=1}^m \omega^i$, and $\theta^0 = \sum_{i=1}^m \theta^i$; let $M_1 = 2\|\omega^0\|_1$, $\bar{X}^i = \{x^i \in X \mid \|x^i\|_\infty \le M_1\}$, and $\Theta^i = \{\theta^i \in R^J \mid \|\theta^i\|_\infty < \tau\}$, where $\|\cdot\|_1$ denotes the 1-norm of a vector. Let $V_0(p) = \sum_{s=1}^S V_s(p_s)$; noting that $V_0(p)$ is continuous in $p$, we may assume

$$M_2 = \max_{\|p\|_1 = 1} \|V_0(p)\|_\infty.$$

With this let

$$\Delta = \{(p, q) \in R_+^{(S+1)L+J} \mid \|p\|_1 = 1, \|q\|_\infty \le 2M_2\}.$$

Finally let $y^i = (x^i, \theta^i)$, $y = (y^1, \ldots, y^m)$, $\bar{Y}^i = \bar{X}^i \times \Theta^i$, and $\bar{Y} = \prod_{i=1}^m \bar{Y}^i$. Clearly, $\Delta$ and all $\bar{Y}^i$ are convex and compact.

For each $(p, q) \in \Delta$, define

$$\gamma^i(p, q) = \left\{y^i \in \bar{Y}^i \mid y^i \in \operatorname{int}(B^i(p, q) \cap \bar{Y}^i)\right\},$$

where $\operatorname{int}(\cdot)$ denotes interior of a set. For the preferences, note that $\pi_s$ is possible to vanish for some $\pi = (\pi_1, \ldots, \pi_S) \in \Pi^i$, and therefore $\succ_i$ may not be strongly monotone. To deal with this issue we define a sequence of strongly monotone preferences which converges to $\succ_i$. Specifically set

$$V_{\pi,r}^i(x^i) = U_\pi^i(x^i) + r\|x^i\|_1,$$

with $r$ being a positive scalar, and define on $\bar{X}^i$ a preference $\succ_{i,r}$:

$$(x^i)' \succ_{i,r} x^i \Leftrightarrow V_{\pi,r}^i((x^i)') > V_{\pi,r}^i(x^i) \quad \text{for all } \pi \in \Pi^i.$$

Denote by $E(r)$ the economy with preferences $\succ_{i,r}$, so that $E(0)$ is our original economy. Define for each $y^i = (x^i, \theta^i)$

$$P_r^i(y^i) = \{(x^i)' \in \bar{X}^i \mid (x^i)' \succ_{i,r} x^i\} \times \Theta^i.$$

By means of these new preferences we define correspondences $\xi_r^i : \Delta \times \bar{Y} \to \bar{Y}_i$, $i = 1, \ldots, m$, by

$$\xi_r^i(p, q, y) = \begin{cases} \gamma^i(p, q), & \text{if } y^i \notin B^i(p, q), \\ \gamma^i(p, q) \cap P_r^i(y^i), & \text{if } y^i \in B^i(p, q); \end{cases}$$

and $\xi_r^0 : \Delta \times \bar{Y} \to \Delta$, by

$$\xi_r^0(p, q, y) = \left\{(p', q') \in \Delta \mid p' \cdot (x^0 - \omega^0) + (q' - V_0(p')) \cdot \theta^0 > 0\right\}.$$

The first step is to show that every $\xi_r^i$, $i = 0, \ldots, m$, has an open graph. It then follows from the fixed point theorem of Gale and Mas-Colell (1975) that there exists $(p_r, q_r, y_r) \in \Delta \times \bar{Y}$ satisfying $\xi_r^i(p_r, q_r, y_r) = \emptyset$, $i = 0, \ldots, m$.

The second step is to show that $(p_r, q_r, y_r)$ constitutes an equilibrium for $E(r)$. Using the strong monotonicity of $\gtrsim_{i,r}$ and $\xi_r^i(p_r, q_r, y_r) = \emptyset$, this follows from a more or less standard argument. The last step is to show that $(p_r, q_r, y_r)$ converges to an equilibrium of $E(0)$ as $r \to 0$; which follows essentially from the fact that every $\succ_i$, $i = 1, \ldots, m$, has an open graph. □