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Estimating the common break date in large factor models

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HIGHLIGHTS

- A consistent estimator for the common break date of the factor loadings in large dimensional factor models is proposed.
- A modified consistent estimator is proposed when the number of factors is unknown.
- Simulation results confirm the good performance of the estimator in finite samples.

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ABSTRACT

This paper considers large dimensional factor models with structural breaks in the factor loadings at a common date. A consistent estimator for the break date is proposed. Simulation results confirm its good performance for small and moderate sample sizes.

1. Introduction

Factor models have attracted a lot of attention from both theoretical and applied econometricians in the past decade. However, despite the presence of structural breaks caused by fundamental policy changes or technology progresses, most of the studies rely on the restrictive assumption of constant factor loadings (see Bai and Ng, 2008; Stock and Watson, 2011, for reviews of recent developments).

Breitung and Eickmeier (2011) is the first paper that proposes a formal test for the structural breaks in the factors loadings, and their test is extended by Chen et al. (2014) and Han and Inoue (forthcoming) to improve powers. But none of these papers consider the estimation of the break date. Cheng et al. (2013) propose a penalized estimator to estimate the factors loadings and the number of factors in the presence of structural breaks. An estimator of the break date can be obtained as a byproduct of their estimation procedure, but the consistency of this estimator is not proved.

In this paper, we focus on factor models where the factor loadings have structural breaks at a common date. A consistent estimator for the break date based on the least square loss function is proposed, and its rate of convergence is established. An important empirical issue is that the number of factors is usually unknown and will be over estimated in the presence of breaks. We show that our estimator can be easily modified to remain consistent without knowing the true number of factors.

2. The model and the estimator

Consider the following factor model with a common break at k_0 in the factor loadings:

$$Y_t = \begin{cases} A_0F_t + e_t & t \leq k_0 \\ B_0F_t + e_t & t > k_0 \end{cases} \tag{1}$$

where $Y_t = [Y_{1t}, \ldots, Y_{Nt}]'$, $A_0 = [\alpha_{10}, \ldots, \alpha_{N0}]'$, $B_0 = [\beta_{10}, \ldots, \beta_{N0}]'$, $e_t = [e_{1t}, \ldots, e_{Nt}]'$, and F_t is a $r \times 1$ vector of common factors. We only observe $\{Y_{it}\}$ for $i = 1, \ldots, N$, $t = 1, \ldots, T$, and

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the object of this paper is to estimate the *break date* $\tau_0 = k_0/T$, which is assumed to be a constant as $T \to \infty$. Additionally, assume there is a positive number $\pi \in (0, 1/2)$ such that $\tau_0 \in (\pi, 1-\pi)$. Denote $K_T(\pi) = [T\pi, T(1-\pi)]$. Consider the Least Square (LS) estimator of the break point:

$$\hat{k} = \arg\min_{k \in K_T(\pi)} \left[\min_{F,A,B} S_{NT}(k, F, A, B) \right], \quad \text{and} \quad \hat{\tau} = \hat{k}/T,$$

where

$$S_{NT}(k, F, A, B) = \sum_{t=1}^{k} \left\| Y_t - AF_t \right\|^2 + \sum_{t=k+1}^{T} \left\| Y_t - BF_t \right\|^2. \quad (2)$$

Define $Y_{(k)} = [Y_1, \ldots, Y_k]'$, $Y^*_{(k)} = [Y_{k+1}, \ldots, Y_T]'$, $F_{(k)} = [F_1, \ldots, F_k]'$ and $F^*_{(k)} = [F_{k+1}, \ldots, F_T]'$. It is well known (see Bai and Ng, 2002, for example) that

$$\min_{F,A,B} S_{NT}(k, F, A, B) = \sum_{t=1}^{T} Y_t' Y_t - \sum_{j=1}^{r} \rho_j \left[Y_{(k)} Y_{(k)}' \right] - \sum_{j=1}^{r} \rho_j \left[Y^*_{(k)} Y^{*'}_{(k)} \right]$$

subject to $k^{-1} F_{(k)}' F_{(k)} = I_r$ and $(T-k)^{-1} F^{*'}_{(k)} F^*_{(k)} = I_r$, where $\rho_j[Q]$ denotes the jth largest eigenvalue of matrix Q. Since $\sum_{t=1}^{T} Y_t' Y_t$ does not depend on k, we can write

$$\hat{k} = \arg\max_{k \in K_T(\pi)} V_{NT}(k), \qquad \hat{\tau} = \hat{k}/T, \quad (3)$$

where

$$V_{NT}(k) = (NT)^{-1} \sum_{j=1}^{r} \Big(\rho_j \left[Y_{(k)} Y_{(k)}' \right] + \rho_j \left[Y^*_{(k)} Y^{*'}_{(k)} \right] \Big).$$

To establish the consistency of the LS estimator, we need to impose the following assumptions:

Assumption 1 (*Factors and Factor Loadings*). (a) $T^{-1} \sum_{t=1}^{T} F_t F_t' \xrightarrow{p} \Sigma_F > 0$ as $T \to \infty$. $T^{-1} \sum_{t=1}^{k} F_t F_t'$ and $T^{-1} \sum_{t=k}^{T} F_t F_t'$ have full rank for large T and for all $k \in K_T(\pi)$. (b) $\sup_{k \in K_T(\pi)} \left\| k^{-1} \sum_{t=1}^{k} F_t F_t' - \Sigma_F \right\| = O_p(T^{-1/2})$. (c) $\left\| N^{-1} A_0' A_0 - \Sigma_A \right\| = O(N^{-1/2})$ for some $\Sigma_A > 0$ as $N \to \infty$, and $N^{-1} A_0' A_0$ have full rank for large N. (d) $E \| N^{-1/2} \sum_{i=1}^{N} \alpha_{i0} e_{it} \|^2 < \infty$ and $E \| N^{-1/2} \sum_{i=1}^{N} \beta_{i0} e_{it} \|^2 < \infty$ for all t.

Assumption 2 (*Idiosyncratic Errors*). (a) Let $\sigma_{N,ts} = N^{-1} \sum_{i=1}^{N} E(e_{it} e_{is})$, then $\sup_t \sum_{s=1}^{T} |\sigma_{N,ts}| \leq \infty$. (b) $\sup_{t,s} N^{-1} \sum_{i,j=1}^{N} \left| Cov\left[e_{it} e_{is}, e_{jt} e_{jt} \right] \right| \leq \infty$.

Assumption 3 (*Breaks*). Let $C_0 = [A_0, B_0]$, and $\left\| N^{-1} C_0' C_0 - \Sigma_C \right\| = O(N^{-1/2})$ and $\Sigma_C \geq 0$.

3. The main results

In this section, we first establish the consistency of $\hat{\tau}$, and then propose a modified consistent estimator for τ when r is unknown. To simplify the notations, in the rest of the paper, we use $A, B, C, \alpha_i, \beta_i$ instead of $A_0, B_0, C_0, \alpha_{i0}, \beta_{i0}$.

Let $\tau = k/T$ and $\delta_{N,T} = \min\{\sqrt{N}, \sqrt{T}\}$, we can prove the following important result, which gives the probability limit of the object function.

Theorem 1. *Under Assumptions 1–3,*

$$\sup_{k \in K_T(\pi)} |V_{NT}(k) - V(k/T)| = O_p(\delta_{N,T}^{-1}),$$

where for $\pi \leq \tau \leq 1-\pi$

$$V(\tau) = \begin{cases} \sum_{j=1}^{r} \Big(\tau \cdot \rho_j[\Sigma_A \Sigma_F] + \rho_j[D_{1\tau}] \Big) & \text{for } \tau \leq \tau_0; \\ \sum_{j=1}^{r} \Big((1-\tau) \cdot \rho_j[\Sigma_B \Sigma_F] + \rho_j[D_{2\tau}] \Big) & \text{for } \tau > \tau_0. \end{cases}$$

The matrices appearing in the definition of $V(\tau)$ are given as follows:

$$\Sigma_{AB} = \lim_{N \to \infty} N^{-1} A' B, \qquad \Sigma_B = \lim_{N \to \infty} N^{-1} B' B,$$

$$D_{1\tau} = \begin{pmatrix} (\tau_0 - \tau) \Sigma_A \Sigma_F & \sqrt{\tau_0 - \tau} \sqrt{1-\tau_0} \Sigma_{AB} \Sigma_F \\ \sqrt{\tau_0 - \tau} \sqrt{1-\tau_0} \Sigma_{AB}' \Sigma_F & (1-\tau_0) \Sigma_B \Sigma_F \end{pmatrix},$$

$$D_{2\tau} = \begin{pmatrix} \tau_0 \Sigma_A \Sigma_F & \sqrt{\tau_0(\tau - \tau_0)} \Sigma_{AB} \Sigma_F \\ \sqrt{\tau_0(\tau - \tau_0)} \Sigma_{AB}' \Sigma_F & (\tau - \tau_0) \Sigma_B \Sigma_F \end{pmatrix}.$$

From the definitions of $D_{1\tau}$ and $D_{2\tau}$ we have

$$V(\tau_0) = \sum_{j=1}^{r} \Big(\tau_0 \cdot \rho_j[\Sigma_A \Sigma_F] + (1-\tau_0) \cdot \rho_j[\Sigma_B \Sigma_F] \Big)$$

and

$$V(\tau_0) - V(\tau) = \begin{cases} \mathrm{Tr}(D_{1\tau}) - \sum_{j=1}^{r} \rho_j[D_{1\tau}] = \sum_{j=r+1}^{2r} \rho_j[D_{1\tau}] & \text{for } \tau \leq \tau_0; \\ \mathrm{Tr}(D_{2\tau}) - \sum_{j=1}^{r} \rho_j[D_{2\tau}] = \sum_{j=r+1}^{2r} \rho_j[D_{2\tau}] & \text{for } \tau > \tau_0. \end{cases} \quad (4)$$

3.1. When r is known

Due to symmetry, we only need to consider $\tau \leq \tau_0$, and it is without loss of generality to assume $\Sigma_F = I_r$. Since $D_{1\tau}$ is a semi-positive definite matrix for any τ, it follows from (4) that $V(\tau_0) - V(\tau) \geq 0$, and the equality holds only when $\sum_{j=r+1}^{2r} \rho_j[D_\tau] = 0$.

First, consider the case where $\mathrm{rank}(\Sigma_C) = r$. It follows that $\mathrm{rank}(D_{1\tau}) = r$ and thus $\sum_{j=r+1}^{2r} \rho_j[D_{1\tau}] = 0$ for all τ. Therefore, we have $V(\tau) = V(\tau_0)$ for all $\tau \in [\pi, \tau_0]$, and the date of such breaks cannot be consistently estimated.

Second, consider the case where $\mathrm{rank}(\Sigma_C) > r$. It is clear that now $\mathrm{rank}(D_{1\tau}) > r$, and $\sum_{j=r+1}^{2r} \rho_j[D_{1\tau}] = 0$ holds only when $\tau = \tau_0$. Thus, we have shown that the object function converges uniformly to $V(\tau)$, which is maximized uniquely at τ_0. It then follows from standard result of extremum estimators (see Newey and McFadden, 1994, for example) that $\hat{\tau} - \tau_0 = o_p(1)$.

To establish the rate of convergence for $\hat{\tau}$, observe that

$$V_{NT}(k) - V_{NT}(k_0) = V_{NT}(k) - V(\tau) - (V_{NT}(k_0) - V(\tau_0)) + V(\tau) - V(\tau_0), \quad (5)$$

and by the definition of $\hat{k}$ and $\hat{\tau}$ we have $V_{NT}(\hat{k}) - V_{NT}(k_0) \geq 0$ for all $k \in K_T(\pi)$. Then it follows from (5) that

$$0 \leq V(\tau_0) - V(\hat{\tau}) \leq 2 \Bigg(\sup_{k \in K_T(\pi)} |V_{NT}(k) - V(\tau)| \Bigg). \quad (6)$$

In Appendix, we show that (Lemma 3) when $\mathrm{rank}(\Sigma_C) > r$, $V(\tau_0) - V(\tau) \geq C_0 |\tau_0 - \tau|$ for some $C_0 > 0$, which depends on Σ_C and τ_0 but not τ. The following result is therefore a direct consequence of (6), Theorem 2, and Lemma 3: