Economics Letters 136 (2015) 42–45

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

About the hyperbolic Lorenz curve

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HIGHLIGHTS

- Wang and Smyth (2015) propose a bi-parametric Lorenz curve.
- We demonstrate that it is a reparameterization of the hyperbolic Lorenz curve.
- We provide two different estimators for the parameters of the model.
- We obtain closed expressions for several inequality measures.

ARTICLE INFO

Article history:
Received 6 August 2015
Received in revised form
4 September 2015
Accepted 7 September 2015
Available online 11 September 2015

JEL classification:
C80
D30

Keywords:
Lorenz curve
Donaldson–Weymark–Kakwani index
Pietra index

ABSTRACT

In a recent paper in this journal, Wang and Smyth (2015) propose a new bi-parametric functional form for the Lorenz curve and use it to derive new parametric forms. In this paper, we demonstrate that the new bi-parametric model is a reparameterization of the hyperbolic Lorenz curve proposed by Arnold (1986). We obtain new and important properties not previously considered.

1. Introduction

In a recent paper in this journal, Wang and Smyth (2015) propose a new bi-parametric functional form for the Lorenz curve (LC) which is used to obtain new parametric forms with different curvatures and introduce a methodology to create convex combination models. These convex combinations were used to build new parametric versions of the LCs.

The objectives of this paper are two: (a) To demonstrate that the model proposed by Wang and Smyth (2015) is a reparameterization of the model proposed by Arnold (1986) and (b) to establish new and important economic properties for this model.

The contents of this paper are as follows. In Section 2 we introduce the models for the LC. The Lorenz ordering and several inequality measures are studied in Sections 3 and 4. Estimation methods are discussed in Section 5. Finally, an empirical application is provided in Section 6.

2. Models of Lorenz curves

The bi-parametric LC proposed by Wang and Smyth (2015) is defined by,

$$L_W(p;\omega_1,\omega_2)=\frac{1-\omega_1}{1+\omega_2}\cdot\frac{(1+\omega_2 p)p}{1-\omega_1 p},\tag{1}$$

where $\omega_1<1$, $\omega_2\geq-\omega_1$, with $0\leq p\leq 1$. The feasible range of the parameters is,

$$\Omega=\{(\omega_1,\omega_2):\ \omega_1<1,\ \omega_2\geq-\omega_1\}.$$

Arnold (1986) considered the hyperbolic LC defined as,

$$L_A(p;\eta,\delta)=\frac{p(1+(\eta-1)p)}{1+(\eta-1)p+\delta(1-p)},\tag{2}$$

where $\eta>0$, $\delta>0$ and $\delta-\eta+1>0$.

If in expression (2) we define:

$$\delta=\frac{\omega_1+\omega_2}{1-\omega_1},$$

$$\eta=1+\omega_2,$$

we obtain (1).

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http://dx.doi.org/10.1016/j.econlet.2015.09.005
0165-1765/

The LC (1) can be interpreted as a linear combination of an infinite set of power LCs $L_1(p) = p^{k+1}$ and $L_2(p) = p^{k+2}$, with weights $\pi_k^{(i)}$, $i = 1, 2$, $k = 0, 1, 2, \ldots$, that is (expanding in power series):

$$L_W(p; \omega_1, \omega_2) = \sum_{k=0}^{\infty} (\pi_k^{(1)} p^{k+1} + \pi_k^{(2)} p^{k+2}),$$

where:

$$\pi_k^{(1)} = \frac{(1-\omega_1)\omega_1^k}{1+\omega_2},$$

and

$$\pi_k^{(2)} = \frac{(1-\omega_1)\omega_2\omega_1^k}{1+\omega_2},$$

with $\sum_{k=0}^{\infty}(\pi_k^{(1)} + \pi_k^{(2)}) = 1$.

From model (1), we consider the following three relevant submodels (see Wang and Smyth, 2015):

(i) If $\omega_1 = 0$, we get ($\omega_2 \geq 0$),

$$L_1(p; \omega_2) = \frac{1}{1+\omega_2} p + \frac{\omega_2}{1+\omega_2} p^2, \tag{3}$$

which is a finite mixture of the egalitarian and the quadratic LCs, with weights $\pi = \frac{1}{1+\omega_2}$ and $1 - \pi = \frac{\omega_2}{1+\omega_2}$.

(ii) If $\omega_2 = 0$, we obtain:

$$L_2(p; \omega_1) = \frac{(1-\omega_1)p}{1-\omega_1 p}, \tag{4}$$

where $0 < \omega_1 < 1$, which is the LC considered by Aggarwal (1984), Arnold (1986), Rohde (2009) and Sarabia et al. (2010). The Aggarwal (1984) curve is included in the Arnold's (1986) class as a special self-symmetric case.

(iii) If $\omega_1 = 2\omega_2 = \delta$, we obtain

$$L_3(p; \delta) = \frac{1-\delta}{2+\delta} \cdot \frac{(2+\delta p)p}{1-\delta p}, \tag{5}$$

with $0 < \delta < 1$.

3. Lorenz ordering

Let $\mathcal{L}$ be the class of all non-negative random variables with positive finite expectation. The Lorenz partial ordering $\preceq_L$ on the class $\mathcal{L}$ is defined by,

$$X \preceq_L Y \iff L_X(p) \geq L_Y(p), \quad \forall p \in [0, 1].$$

If $X \preceq_L Y$, then X exhibits less inequality than Y in the Lorenz sense. We shall show that family (1) is ordered with respect parameters ω_1 and ω_2.

Lemma 1. *Let $L_W(p; \omega_1, \omega_2)$ the LC defined in (1). If $(\omega_1^{(i)}, \omega_2) \in \Omega$, $i = 1, 2$ with $\omega_1^{(1)} \geq \omega_1^{(2)}$ and $1+\omega_2 > 0$ then $L_W(p; \omega_1^{(1)}, \omega_2) \leq L_W(p; \omega_1^{(2)}, \omega_2)$, for $0 \leq p \leq 1$. On the other hand, if $(\omega_1, \omega_2^{(i)}) \in \Omega$, $i = 1, 2$ with $\omega_2^{(1)} \geq \omega_2^{(2)}$ and $0 < \omega_1 < 1$ then $L_W(p; \omega_1, \omega_2^{(1)}) \leq L_W(p; \omega_1, \omega_2^{(2)})$, for $0 \leq p \leq 1$.*

Proof. Direct, taking partial derivatives in (1) with respect to ω_i, $i = 1, 2$. ■

4. Inequality measures

A relevant generalization of the Gini index was proposed by Donaldson and Weymark (1980), Kakwani (1980) and studied in detail by Yitzhaki (1983). These authors proposed the generalized Gini index defined as,

$$G_X(\nu) = 1 - \nu(\nu+1)\int_0^1 (1-p)^{\nu-1} L_X(p)dp, \tag{6}$$

where $\nu > 1$ and $L_X(\cdot)$ is the LC. If we set $\nu = 1$ in Eq. (6) we obtain the Gini index. As ν increases, more weight is attached to the lower tail of the distribution. In the limit when ν goes to infinity the index only considers the minimum income, which is congruent with the Rawlsian criterion, expressing the judgement that social welfare depends only on the poorest society member.

The Donaldson–Weymark–Kakwani (DWK) index for the LC (3) is,

$$G_W(\omega_2) = 1 - \frac{(1+\nu)(2+\nu+2\omega_2)}{(2+3\nu+\nu^2)(1+\omega_2)}.$$

For the model (4) this index was obtained by Sarabia et al. (2010, equation (4)). The following theorem provides the DWK index for the general case.

Theorem 1. *The DWK index for the LC defined in (1) is given by Eq. (7) in Box I, where $H_1 = {}_2F_1[1, 1; 2+\nu; \omega_1]$, $H_2 = {}_2F_1[1, 1; 3+\nu; \omega_1]$, and ${}_2F_1[a, b; c; z]$ represent the hypergeometric function, which can be written in the form,*

$$ {}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \tag{8}$$

where $c > b > 0$.

Proof. Direct, using Eqs. (6) and (8). ■

We consider two particular cases of formula (7):

- $\nu = 1$ (Gini index):

$$G_W(1) = \frac{(\omega_1+\omega_2)(2\omega_1 - \omega_1^2 + 2(1-\omega_1)\log(1-\omega_1))}{\omega_1^3(1+\omega_2)}, \tag{9}$$

- $\nu = 2$:

$$G_W(2) = \frac{(\omega_1+\omega_2)(-6\omega_1 + 9\omega_1^2 - 2\omega_1^3 - 6(1-2\omega_1-\omega_1^3)\log(1-\omega_1))}{\omega_1^4(1+\omega_2)}. \tag{10}$$

Formula (9) is equivalent to formula (5) in Wang and Smyth (2015).

The Pietra index is defined as the maximal vertical deviation between the LC and the egalitarian line (see Sarabia and Jordá, 2014),

$$P_L = \max_{0 \leq p \leq 1} \{p - L_X(p)\}.$$

For the special case (3) the Pietra index is $\frac{1}{2}$. For the general LC in (1) we have the following simple expression for the Pietra index:

$$P_W = \frac{1-\sqrt{1-\omega_1}}{\omega_1}, \tag{11}$$

which only depends on ω_1. This means that the Pietra index (that is, the quantity $E|X - \mu_X|/2\mu_X$) does not provide information about all the features of the distribution, in particular those depend on the ω_2 parameter. In contrast, the Gini index depends on the two parameters ω_1 and ω_2. Note that the Pietra index is insensitive to transfers on one side of the mean, while the Gini index is sensitive any Pigou Dalton transfer.