Economics Letters 136 (2015) 95–98

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Monotonic redistribution of non-negative allocations: A case for proportional taxation revisited☆

André Casajus

Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

HIGHLIGHTS

- We reconsider Casajus' (TE, forthcoming) characterization of proportional taxation.
- He characterizes taxation at a fixed rate by efficiency, symmetry, and monotonicity.
- This does not work on the restricted domain of non-negative income.
- On this domain, the tax rate may vary with total income.
- There are upper and lower bounds on the elasticity of the tax rate function.

ARTICLE INFO

Article history:
Received 15 June 2015
Received in revised form
27 August 2015
Accepted 11 September 2015
Available online 24 September 2015

JEL classification:
C71
D63
H20

MSC:
91A12
91B15

Keywords:
Redistribution
Proportional taxation
Efficiency
Symmetry
Monotonicity

ABSTRACT

We reconsider Casajus' (forthcoming) characterization of uniformly proportional taxation by three properties of redistribution: efficiency, symmetry, and monotonicity. When restricted to non-negative income, these properties imply proportional taxation in a weaker sense—the tax rate may vary with total income but only in an economically reasonable way.

> And all the tithe of the land, whether of the seed of the land, or of the fruit of the tree, is the Lord's: it is holy unto the Lord.
> — Leviticus 27:30

1. Introduction

The above quotation from the King James Bible provides an early example of the idea of proportional taxation in a strong sense—the tax rate neither depends on individual income nor on the total income of the society. Ju et al. (2007, Remarks on Theorem 7) provide a nice axiomatic justification of this kind of taxation[1] within the standard framework of the redistribution of non-negative income.[2]

Recently, Casajus (forthcoming) supports such proportional taxation by requiring three properties for the redistribution of income, two standard properties, efficiency and symmetry, and a

☆ We are grateful to Frank Huettner and Hervé Moulin as well as to an anonymous referee for helpful comments on this paper. Financial support by the Deutsche Forschungsgemeinschaft, grant CA 266/4-1 is gratefully acknowledged.

E-mail address: mail@casajus.de.
URL: http://www.casajus.de.

[1] Fleubaey and Maniquet (2011, Chapters 10 and 11), for example, provide a survey of axiomatic foundations of taxation in general.

[2] Lambert (2002), for example, provides an overview on the formal treatment of redistribution.

http://dx.doi.org/10.1016/j.econlet.2015.09.015

monotonicity property: an individual's income after redistribution must not decrease whenever both her income and the total income of the society do not decrease. A peculiarity of Casajus' result, however, is that it applies to the redistribution of gains and losses, i.e., income may be negative.

In this note, we explore whether Casajus' result extends to the restricted domain. Within the restricted domain, his properties only imply a weaker version of proportional taxation (Theorem 3). Redistribution takes place by proportional taxation, where the tax rate may vary with total income but only in an economically reasonable way. Technically, this can be expressed by lower and upper bounds on the elasticity of the tax rate function (Theorem 5).

The next section gives a formal account and discussion of these results. Some remarks conclude the paper. Two appendices contain the lengthier proofs of our results.

2. Monotonic redistribution rules and proportional taxation

Casajus (forthcoming) considers redistribution in a simple model of a society. For $n \in \mathbb{N}$, the members of an n-person society are represented by the first n natural numbers; $\mathbb{N}_n := \{1, \ldots, n\}$; individual incomes are given by a vector $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$, we set $\bar{x} := \sum_{\ell \in \mathbb{N}_n} x_\ell$. A **redistribution rule** for an n-person society is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $i \in \mathbb{N}_n, f_i(x)$ denotes the income of member i of the society after redistribution. The properties of redistribution rules mentioned in the introduction are formally defined as follows.

Efficiency, E. For all $x \in \mathbb{R}^n$, we have $\sum_{\ell \in \mathbb{N}_n} f_\ell(x) = \bar{x}$.

Symmetry, S. For all $x \in \mathbb{R}^n$ and $i, j \in \mathbb{N}_n$ such that $x_i = x_j$, we have $f_i(x) = f_j(x)$.

Monotonicity, M. For all $x, y \in \mathbb{R}^n$ and $i \in \mathbb{N}_n$ such that $\bar{x} \geq \bar{y}$ and $x_i \geq y_i$, we have $f_i(x) \geq f_i(y)$.

For societies with more than two members, these properties already imply uniformly proportional taxation.[3] Individual incomes are taxed at a fixed rate and overall tax revenue is distributed equally among the society's members.[4]

Theorem 1 (*Casajus, forthcoming*)**.** *Let $n > 2$. A redistribution rule $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies efficiency (**E**), symmetry (**S**), and monotonicity (**M**) if and only if there exists some $\tau \in [0, 1]$ such that*

$$f_i(x) = (1 - \tau) \cdot x_i + \frac{\tau \cdot \bar{x}}{n} \quad \text{for all } x \in \mathbb{R}^n \text{ and } i \in \mathbb{N}_n. \tag{1}$$

The proof of this result, in particular, the proof that the tax rate does not depend on the total income makes use of an unbounded-domain assumption, i.e., the fact that we consider arbitrary great or small (negative) individual incomes. Since the interpretation of the taxation of negative income is less convincing than for non-negative income, one might wonder whether Theorem 1 remains true if one restricts attention to non-negative income. That is, the domain of redistribution rules is $\mathbb{R}^n_+$.[5] Moreover, the properties are required to hold only for $x, y \in \mathbb{R}^n_+$. We indicate this by a subscript "+" at the abbreviations of the properties.

It turns out that Casajus' characterization does not work when restricted to non-negative income. A counterexample can be found in Casajus (2015, Theorem 1). This triggers the question of which type of redistribution is implied by the restricted properties.

This question is partly answered by Ju et al. (2007, Theorem 7). They consider the following properties of redistribution rules.

Reallocation-proofness, RP. For all $x, y \in \mathbb{R}^n_+$ and $S \subseteq \mathbb{N}_n$ such that $x_i = y_i$ for all $i \in \mathbb{N}_n \setminus S$ and $\sum_{i \in S} x_i = \sum_{i \in S} y_i$, we have $\sum_{i \in S} f_i(x) = \sum_{i \in S} f_i(y)$.

Non-negativity, NN. For all $x \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$, we have $f_i(x) \geq 0$.

No transfer paradox, NTP. For all $x, y \in \mathbb{R}^n_+$ and $i, j \in \mathbb{N}_n, i \neq j$ such that $x_k = y_k$ for all $k \in \mathbb{N}_n \setminus \{i, j\}$, $x_i + x_j = y_i + y_j$, and $x_i \geq y_i$, we have $f_i(x) \geq f_i(y)$.

Combined with efficiency, the three properties imply that income is redistributed by taxation and giving each member of the society a certain share of the total tax revenue, where both the tax rate and the individual shares may vary with the total income in the society without any restrictions.

Theorem 2 (*Ju et al., 2007*)**.** *Let $n > 2$. A redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies efficiency (**E_+**), reallocation-proofness (**RP**), non-negativity (**NN**), and no transfer paradox (**NTP**) if and only if there exist two mappings $\tau : \mathbb{R}_{++} \to [0, 1]$ and $\sigma : \mathbb{R}_{++} \to \Delta^n_+$ such that*

$$f_i(x) = \begin{cases} 0, & \bar{x} = 0, \\ (1 - \tau(\bar{x})) \cdot x_i + \sigma_i(\bar{x}) \cdot \tau(\bar{x}) \cdot \bar{x}, & \bar{x} > 0, \end{cases}$$
$$\text{for all } x \in \mathbb{R}^n_+ \text{ and } i \in \mathbb{N}_n. \tag{2}$$

This result is related to our question as follows. First, if a redistribution rule as in (2) is required to be symmetric, then all members of the society have to obtain the same share of the total tax revenue. Second, it is straightforward to show that monotonicity implies the no transfer paradox, that monotonicity and efficiency imply reallocation-proofness, and that monotonicity, symmetry, and efficiency imply non-negativity.

The complete answer to our question is given by the next theorem. Its proof is referred to Appendix A. Note that the restriction of Casajus' counterexample for $n = 2$ also works on the domain of non-negative allocations.

Theorem 3. *Let $n > 2$. A redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies efficiency (**E_+**), symmetry (**S_+**), and monotonicity (**M_+**) if and only if there is a mapping $\tau : \mathbb{R}_{++} \to [0, 1]$ with the following properties.*

(i) *For all $x \in \mathbb{R}^n_+$ and $i \in \mathbb{N}_n$, we have*

$$f_i(x) = \begin{cases} 0, & \bar{x} = 0, \\ (1 - \tau(\bar{x})) \cdot x_i + \dfrac{\tau(\bar{x}) \cdot \bar{x}}{n}, & \bar{x} > 0. \end{cases} \tag{3}$$

(ii) *For all $c, d \in \mathbb{R}_{++}$ such that $d \geq c$, we have $\tau(d) \cdot d \geq \tau(c) \cdot c$.*

(iii) *For all $c, d \in \mathbb{R}_{++}$ such that $d \geq c$, we have*

$$\frac{\tau(d) \cdot d}{n} - \frac{\tau(c) \cdot c}{n} \geq (\tau(d) - \tau(c)) \cdot c.$$

While Ju et al. (2007, Theorem 7) impose no restrictions on the tax rate function, Casajus (forthcoming, Theorem 1) requires the tax rate to be fixed for all total incomes. Theorem 3 steers a middle course. The tax rate vary with the total income but in an economically reasonable way.

By condition (ii), overall tax revenue cannot decrease with increasing total income, i.e., the tax rate is not allowed to drop too fast with increasing total income. Since members of the society with a zero income obtain a fraction of overall tax revenue, this property "protects" the weakest members of the society.

Given that condition (ii) holds true, condition (iii) is always satisfied when the tax rate decreases. Hence, condition (iii) requires the tax rate not to increase too much with increasing total income. In particular, overall tax paid (on a given total

3 Throughout the paper, we disregard the trivial case $n = 1$.

4 Casajus and Huettner (2014) extend this result to one-point solutions of cooperative games with transferable utility, in a sense.

5 We set $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$, and $\Delta^n_+ := \{s \in \mathbb{R}^n_+ \mid \bar{s} = 1\}$.