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Limit theory for an explosive autoregressive process[☆]

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HIGHLIGHTS

- Establish large sample properties for AR(1) with an intercept and an explosive root.
- Show that the LS estimate of intercept and its t-statistic are asymptotically normal.
- Show that no invariance principle applies to autoregressive coefficient estimate.
- Show that tests have better power for the zero intercept in the explosive case.

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ABSTRACT

Large sample properties are studied for a first-order autoregression (AR(1)) with a root greater than unity. It is shown that, contrary to the AR coefficient, the least-squares (LS) estimator of the intercept and its t-statistic are asymptotically normal without requiring the Gaussian error distribution, and hence an invariance principle applies. The coefficient based test and the t test have better power for testing the hypothesis of zero intercept in the explosive process than in the stationary process.

1. Introduction

Consider a first-order autoregression defined by

$$x_t = d + \alpha x_{t-1} + u_t, \quad x_0 \sim O_p(1), \tag{1.1}$$

where u_t is a sequence of independent and identically distributed (i.i.d.) random errors with $E(u_t) = 0, E\left(u_t^2\right) = \sigma^2 \in (0, \infty)$ (i.e., $u_t \overset{iid}{\sim} (0, \sigma^2)$). The available sample is $\{x_t\}_{t=1}^T$. Let $\sum$ denote $\sum_{t=1}^T$. If d is known a priori and assumed zero without loss of generality, based on the available sample, the least-squares (LS) estimator of α is,

$$\widehat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}. \tag{1.2}$$

If the value of d is unknown a priori, the LS estimators of α and d are, respectively,

$$\widehat{\alpha} = \frac{\sum \left(x_t - \overline{X}\right)\left(x_{t-1} - \overline{X}_-\right)}{\sum \left(x_{t-1} - \overline{X}_-\right)^2} \quad \text{and} \quad \widehat{d} = \overline{X} - \widehat{\alpha}\overline{X}_-, \tag{1.3}$$

where $\overline{X} = \sum x_t / T, \overline{X}_- = \sum x_{t-1}/T$.

The limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics have been developed in the literature in several special cases of Model (1.1), including the stationary case ($|\alpha| < 1$), the unit root case ($\alpha = 1$), and the explosive case ($|\alpha| > 1$). Hamilton (1994) provides the textbook treatment of the unit root case in the page range 490–494 and the stationary case in page 216.

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If $|\alpha| > 1, x_0 = 0, u_t \overset{iid}{\sim} N(0, \sigma^2)$, and $d = 0$ and is known a priori, White (1958) showed that

$$\frac{\alpha^T}{\alpha^{2-1}}(\widehat{\alpha} - \alpha) \Rightarrow Cauchy.$$

When $u_t \overset{iid}{\sim}(0, \sigma^2)$ but is not necessarily normally distributed, Anderson (1959) showed that

$$\frac{\alpha^T}{a^2 - 1}(\widehat{\alpha} - \alpha) \Rightarrow y/z,$$

where y and z are the limits of y_T and z_T defined by

$$y_T = \sum_{t=1}^{T} \alpha^{-(T-t)} u_t \quad \text{and} \quad z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0. \tag{1.4}$$

Obviously the limiting distributions of y_T and z_T, and hence of $\widehat{\alpha}$, depend on the distribution of u's, so no central limit theorem (CLT) or invariance principle is applicable. The role played by the initial condition in the limiting distribution could be found in z. In this case the rate of the convergence depends on both T and α.

In this paper, we extend the literature by establishing the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics for the explosive AR(1) process with an unknown intercept. We show that the asymptotic normality and, hence, an invariance principle hold true for $\widehat{d}$ and its t-statistic without assuming the Gaussian error distribution. The motivation for our study comes from a recent literature on econometric analysis of bubbles; see for example, Phillips et al. (2011, 2014, 2015a, forthcoming, 2015b, forthcoming). All proofs are in the Appendix.

2. The model

We now focus our attention on Model (1.1) with $|\alpha| > 1$. An equivalent representation of x_t is

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + \alpha^t x_0 + \sum_{j=0}^{t-1} \alpha^j u_{t-j}. \tag{2.1}$$

Obviously, $(1 - \alpha^t) d/(1 - \alpha)$ and $\alpha^t x_0$ have the same order of $O_p(\alpha^t)$ if $d \neq 0$. It becomes clear later that $\sum_{j=0}^{t-1} \alpha^j u_{t-j}$ has the order of $O_p(\alpha^t)$ too. This is the reason why both the intercept and the initial condition play an important role in the asymptotic theory for the explosive process. The model can also be expressed as

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + x_t^0, \tag{2.2}$$

where x_t^0 is an explosive AR(1) process with no intercept.

Denote

$$w_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t. \tag{2.3}$$

Following the Lindeberg–Feller CLT, the limiting distribution of w_T is $N(0, \sigma^2)$. Following Anderson (1959), we define y_T and z_T as in Eq. (1.4). In the following lemma whose proof is in the Appendix, we give the limits of w_T, y_T, and z_T, and show that they are independent from each other.

Lemma 2.1. *Define w_T, y_T, and z_T as in Eqs. (1.4) and (2.3). Then we have* (a) $y_T \Rightarrow y, z_T \Rightarrow z$, *and y and z are independent;* (b) $w_T \Rightarrow w \overset{d}{=} N(0, \sigma^2)$ *and w is independent of (y, z).*

To obtain the limiting distribution of the LS estimator of α in the explosive AR(1) model without intercept, Anderson (1959) proved that

$$\left(\alpha^{-(T-2)} \sum x_{t-1}^0 u_t,\ (\alpha^2 - 1) \alpha^{-2(T-1)} \sum (x_{t-1}^0)^2\right) \Rightarrow (yz, z^2). \tag{2.4}$$

Using this result together with the independence of w, y, z, we obtain the following results.

Theorem 2.2. *For Model (1.1) with $|\alpha| > 1$, we have, as $T \to \infty$,*

(a) $\alpha^{-(T-1)} x_T \Rightarrow z + \alpha d/(\alpha - 1)$;
(b) $\alpha^{-(T-2)} \sum x_{t-1} u_t \Rightarrow y[z + \alpha d/(\alpha - 1)]$;
(c) $(\alpha - 1) \alpha^{-(T-1)} \sum x_{t-1} \Rightarrow z + \alpha d/(\alpha - 1)$;
(d) $(\alpha^2 - 1) \alpha^{-2(T-1)} \sum x_{t-1}^2 \Rightarrow [z + \alpha d/(\alpha - 1)]^2$.

Since $z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0$, not surprisingly, the initial condition αx_0 appears in the limit, z. According to Theorem 2.2, the intercept term d appears in all the asymptotic distributions. In particular, the intercept and the initial condition affect the asymptotic distributions in the same manner. This observation is consistent with the one in Eq. (2.1) where the three terms on the right hand side have the same order of magnitude.

The centered LS estimators of d and α and their t-statistics are given by

$$\begin{pmatrix} \widehat{d} - d \\ \widehat{\alpha} - \alpha \end{pmatrix} = \begin{pmatrix} T & \sum x_{t-1} \\ \sum x_{t-1} & \sum x_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum u_t \\ \sum x_{t-1} u_t \end{pmatrix},$$

and

$$t_d = \frac{(\widehat{d} - d)\left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}},$$

$$t_\alpha = \frac{(\widehat{\alpha} - \alpha)\left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[T \times \hat{\sigma}^2\right]^{1/2}},$$

where $\hat{\sigma}^2 = T^{-1} \sum \left(x_t - \hat{d} - \hat{\alpha} x_{t-1}\right)^2$.

Since $\sum x_{t-1} u_t$ and $\sum x_{t-1}$ have the same rate of convergence, α^{-T}, we have

$$\begin{aligned} \begin{pmatrix} \sqrt{T}(\widehat{d} - d) \\ \alpha^T(\widehat{\alpha} - \alpha) \end{pmatrix} &= \begin{pmatrix} 1 & T^{-1/2}\alpha^{-T} \sum x_{t-1} \\ T^{-1/2}\alpha^{-T} \sum x_{t-1} & \alpha^{-2T} \sum x_{t-1}^2 \end{pmatrix}^{-1} \times \begin{pmatrix} T^{-1/2} \sum u_t \\ \alpha^{-T} \sum x_{t-1} u_t \end{pmatrix} \\ &= \begin{pmatrix} 1 & o_p(1) \\ o_p(1) & \alpha^{-2T} \sum x_{t-1}^2 \end{pmatrix}^{-1} \times \begin{pmatrix} T^{-1/2} \sum u_t \\ \alpha^{-T} \sum x_{t-1} u_t \end{pmatrix}. \end{aligned}$$

Consequently, we have the following theorem which extends Anderson's results to the explosive AR(1) model with intercept.

Theorem 2.3. *For Model (1.1) with $|\alpha| > 1$, if $Pr\{z + \alpha d/(\alpha - 1) = 0\} = 0$, the following limits apply as $T \to \infty$:*

(a)

$$\sqrt{T}(\widehat{d} - d) = T^{-1/2} \sum u_t + o_p(1) \Rightarrow w \overset{d}{=} N(0, \sigma^2), \tag{2.5}$$