Economics Letters 129 (2015) 52–56

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

The stability of many-to-many matching with max–min preferences

Zhenhua Jiao [a,b,*], Guoqiang Tian [c]

[a] Institute of Operations Research, Hangzhou Dianzi University, Hangzhou, Zhejiang, 310018, China
[b] School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China
[c] Department of Economics, Texas A&M University, College Station, TX 77843, USA

HIGHLIGHTS

- The equivalence between the pairwise-stability and the setwise-stability is obtained.
- We show that the pairwise-stability implies the strong corewise-stability.
- We show that the strong core may be a proper subset of the core.
- We show that the deferred acceptance algorithm yields a pairwise-stable matching.

ARTICLE INFO

Article history:
Received 6 November 2014
Received in revised form
24 January 2015
Accepted 6 February 2015
Available online 14 February 2015

JEL classification:
C78

Keywords:
Many-to-many matching
Pairwise-stability
Core
Setwise-stability
Max–min preferences

ABSTRACT

This paper investigates the two-sided many-to-many matching problem, where every agent has max–min preference. The equivalence between the pairwise-stability and the setwise-stability is obtained. It is shown that the pairwise-stability implies the strong corewise-stability and the former may be strictly stronger than the latter. We also show that the strong core may be a proper subset of the core. The deferred acceptance algorithm yields a pairwise-stable matching. Thus the set of stable matchings (in all four senses) is non-empty.

1. Introduction

For a matching problem, researchers primarily concern the existence and the structure of stable assignments. There are several concepts on stability of matching: pairwise-stability, corewise-stability, strong corewise-stability and setwise-stability (or group stability). Gale and Shapley (1962, henceforth GS) originally study the stable matching between men and women, and, between students and colleges. They give the definition of (pairwise-)stable matching and propose the deferred acceptance algorithm to yield the optimal stable matching. Roughly speaking, pairwise-stability means that the matching is individually rational and there exists no pair of unmatched players who can become better off if they are matched together. For one-to-one matching problem, it is enough to explore its pairwise-stability. If every player has strict preference, the above-mentioned four different concepts of stability are equivalent for the one-to-one case.

The concepts of the core and the strong core originate from the cooperative game theory. In matching theory, corewise-stability (resp. strong corewise-stability) describes the condition of no subset of players, who by forming *all* their partnerships among themselves, can *all* obtain a strictly preferred set of partners (resp. can *all* obtain a weakly preferred set of partners and *at least one of them* becomes strictly better off). Clearly, the strong corewise-stability strengthens the requirement of the corewise-stability.

Roth (1985) proposes the concept of group-stability in the context of the college admissions problem. For many-to-many matching problem, Sotomayor (1999) called the notion of group-stability as setwise-stability, which characterizes the condition that there is no subset of players who by forming *new* partnerships only among themselves, possibly dissolving some partnerships of the given

* Corresponding author at: School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China. Tel.: +86 571 8687 8525.
E-mail addresses: jzhenhua@163.com (Z. Jiao), gtian@tamu.edu (G. Tian).

http://dx.doi.org/10.1016/j.econlet.2015.02.009
0165-1765/

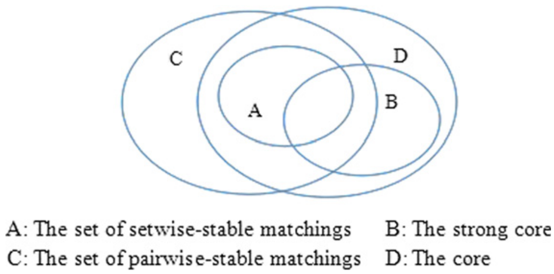

Fig. 1. The relationships between different concepts of stability under separable preference.

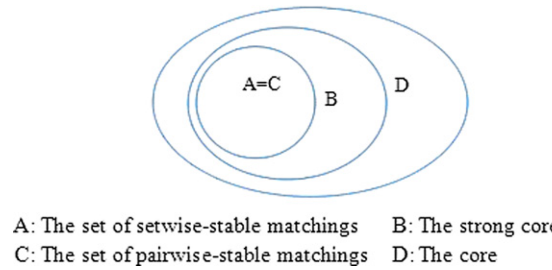

Fig. 2. The relationships between different concepts of stability under max–min preference.

matching to remain within their quotas and possibly keeping other ones, can *all* obtain a strictly preferred set of partners. The concept of setwise-stability generalizes that of pairwise-stability. Obviously, the setwise-stability implies both pairwise-stability and corewise-stability.

Matching theory proceeds normally by imposing hypotheses on agents' preferences. For many-to-many matchings, under the assumption of separable preferences,[1] Sotomayor (1999) shows that pairwise-stability is independent of the corewise-stability and setwise-stability may be strictly stronger than the requirement of pairwise-stability plus corewise-stability. She also constructs an artful example such that both the core and the set of pairwise-stable matchings are nonempty, but their intersection set is empty. Consequently, Sotomayor obtains that there may be no setwise-stable assignment for many-to-many matching with separable preferences. Since separability implies responsiveness and consequently substitutability,[2]the above-mentioned results hold for these three kinds of preferences. More intuitively, the relationship between different concepts of stability obtained by Sotomayor can be expressed by a Venn diagram (see Fig. 1).

This paper investigates the stability of many-to-many matching with max–min preference. For a many-to-many matching problem, we prove that, if every agent has max–min preference, then the deferred acceptance algorithm yields a pairwise-stable assignment. We also show that the pairwise-stability is equivalent to the setwise-stability and the pairwise-stability may be strictly stronger than the strong corewise-stability. Thus it implies that both the core and the strong core are nonempty. Summarily, we obtain the relationship between different concepts of stable matching under max–min preference as in Fig. 2.

2. The model

Let R denote a finite set of *row-players* and C a finite set of *column-players*.[3] Each $r \in R$ has a strict, complete and transitive preference relation $\succ_r$ over C and a quota p_r. The weak preference relation associated with $\succ_r$ is denoted by $\succeq_r$. For any $c_1, c_2 \in C$, $c_1 \succeq_r c_2$ means either $c_1 \succ_r c_2$ or $c_1 = c_2$. The notation $c \succ_r \emptyset$ means that the column player c is *acceptable* to r and $\emptyset \succ_r c$ denotes that c is *unacceptable* to r. For column-players, we can define corresponding notation and denote the quota of column-player c by q_c.

For many-to-many matching, we also need to consider agents' preferences over groups of players on the opposite side. We assume these preferences are transitive, but the completeness is not required. Throughout this paper, we assume that the preference relation of every row-player satisfies the following property:

Weak monotonicity in population: For each row-player r, for any $S \in 2^C$ with $|S| < p_r$ and any column-players c not in S, r prefers $S \cup \{c\}$ to S if and only if c is acceptable to r, where the notation $|S|$ denotes the number of elements in S.[4]

Corresponding weak monotonicity in population for the preference relation of column-player is also required.

Given a set of agent r's partners S, let $\min(S)$ denote the least preferred partner of r in S.

Baïou and Balinski (2000) propose the max–min preference and study the Pareto efficiency and incentives properties for many-to-many matching when every player has max–min preference. In the framework of matching markets and the setting of max–min preference, Hatfield et al. (2014) obtain some negative results on the Pareto efficiency and incentives properties of many-to-many matching by constructing an ingenious example. In this paper, we follow the definition of max–min preference introduced by Baïou and Balinski as below:

Definition 1. The preference relation of row-player $r \in R$ is said to satisfy the *max–min criterion* if the following condition is met: for any two sets of acceptable column-players $S_1, S_2 \in 2^C$ with $|S_1| \le p_r$ and $|S_2| \le p_r$,

(i) The strict preference relation $\succ_r$ over groups of column-players is defined as: $S_1 \succ_r S_2$ if and only if S_2 is a proper subset of S_1 or, $|S_1| \ge |S_2|$ and r strictly prefers the least preferred column-player in S_1 to the least preferred column-player in S_2.
(ii) The weak preference relation $\succeq_r$ over groups of column-players is defined as: $S_1 \succeq_r S_2$ if and only if $S_1 \succ_r S_2$ or $S_1 = S_2$.

The preference relation of column-player $c \in C$ satisfies the max–min criterion if the corresponding condition is met.

We note that there is no implication relationship between max–min preference and responsive preference. In fact, firstly, max–min criterion is not stronger than responsiveness. For example, we assume $c_1 \succ_r c_2 \succ_r c_3$, $p_r = 2$ and $S = \{c_3\}$. Under max–min criterion we cannot achieve $S \cup \{c_1\} \succ_r S \cup \{c_2\}$. Secondly, responsiveness is not stronger than max–min criterion. For example, we assume $c_1 \succ_r c_2 \succ_r c_3 \succ_r c_4$ and $p_r = 2$. Under responsiveness we cannot infer $\{c_2, c_3\} \succ_r \{c_1, c_4\}$.

---

1 Separable preference is defined as follows: let u_{ij} denote the utility that i can get in case i and j form a partnership. For any two sets of partners of i, S, S' with $|S| < q_i$ and $|S'| < q_i$, the player i prefers S to S' if and only if $\sum_{j \in S} u_{ij} > \sum_{j \in S'} u_{ij}$.

2 Responsive preference is defined as follows: For any i, j and any S such that $i, j \notin S$ and $|S| < q_k$, $S \cup \{i\} \succ_k S \cup \{j\}$ if and only if $i \succ_k j$, where i, j are the partners of k and S is a set of partners of k. Separability implies responsiveness because that, for any i, j and any S such that $i, j \notin S$ and $|S| < q_k$, if $\succ_k$ is separable, then by definition it is easy to obtain $S \cup \{i\} \succ_k S \cup \{j\}$ if and only if $i \succ_k j$.

3 For example, row- and column-players may correspond to firms and workers.

4 Row-players' preferences are strongly monotonic in population if $\forall r \in R, \forall S, S' \in 2^C$, $|S'| < |S| \le p_r$ implies $S \succ_r S'$ (see Konishi and Ünver, 2006). Obviously, strong monotonicity implies weak monotonicity.