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Some exact and inexact linear rational expectation models in vector autoregressive models

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HIGHLIGHTS

- Compare exact and inexact linear rational expectation models.
- Characterize the difference.
- Discuss possible elimination to avoid constrained optimization for maximizing likelihood.

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ABSTRACT

In this paper we consider maximum likelihood estimation in some exact and inexact linear rational expectation (LRE) models. The implications of the two models on the coefficients of the vector autoregressive (VAR) model are spelled out. The inexact version is more complicated and possible simplification of the resulting constrained optimization problem is discussed.

1. Introduction

Anticipations about future developments are an important aspect of economic behavior. As one may expect there have therefore been many attempts to incorporate expectations in economic theories. One way to obtain this, when the models contain a stochastic element, is to identify the expectations which are to be modeled with the mathematical conditional expectations given observations up to and including the present ones. This is a commonly used approach which allows economic theories to be formulated taking forward looking behavior into account. Vector autoregressive models (VAR) are one of the work horses of empirical macroeconomics. When the stochastic model is formulated as a VAR, the expectations are one-step ahead and the behavioral restrictions are linear; this will have implications for the coefficients of the VAR.

It turns out that two distinct cases arise. One is the case where the restrictions are exact in the sense that the conditional expectations are completely specified in terms of the other variables, and this will naturally be denoted as the *exact* case. The requirement that the specified relations must hold exactly is in many cases rigid. A natural alternative is to allow for a certain discrepancy described by a random variable. This is the *inexact* case. For a discussion of the two cases one can consult Hansen and Sargent (1991).

In a VAR-model it is reasonable to suppose that the random variables capturing the discrepancy are innovations even if more elaborate specifications can be and often are used. An advantage of the innovation assumption is that the restrictions on the coefficients of the VAR-model can be worked out similarly to the exact case. It is then possible to compare the two specifications. This is interesting since in many cases it is relatively easy to estimate the exact version. For fixed values of the structural parameters it can be done by running some appropriately

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specified regressions; see e.g. Lütkepohl (1991, Section 10.2.3), for the case where the impact matrix has full rank so the process is I(0) and Johansen and Swensen (1999, 2004, 2008) for reduced rank VAR-models. This provides the value of the Gaussian likelihood, conditioned on the initial observations, as a function of the structural parameters, i.e. a concentrated or profile likelihood. Estimates for the structural parameters can then be found by a numerical optimization routine. On the other hand, using maximum likelihood techniques for estimating the inexact specification can be much more computationally demanding. Typically it will involve maximizing over large sets of structural parameters and parameters from the VAR satisfying non linear constraints.

In the present paper we shall compare the restrictions on the VAR coefficients for the two situations. In particular we shall generalize a result of Kurman (2007) showing that non-singularity of the companion matrix is a sufficient condition for an exact model. We also consider in more detail a situation where the maximizing of the likelihood for inexact models, which in general involves maximization under side conditions, can be reduced to maximization without side conditions.

The paper is organized as follows. In Section 2 the VAR-framework is described, and the LRE models formulated and compared. In Section 3 the possibility of reducing the constrained optimization problem to an unconstrained one is discussed.

The following notation will be used: a vector having the j'th element equal to one and the rest equal to zero will be denoted by e_j.

2. General model

The point of departure is p-dimensional observations $X_1, \ldots, X_T$ from a vector autoregressive (VAR) model of order k written as

$$X_t = A_1 X_{t-1} + \cdots + A_k X_{t-k} + \mu_t + \epsilon_t, \tag{1}$$

where $\varepsilon_1, \ldots, \varepsilon_T$ are independent Gaussian variables with expectation zero, unrestricted covariance matrix Ω and μ_t is a non-random sequence. Let the vector μ contain any parameter describing μ_t. Considering the k first observations as fixed the concentrated conditional log likelihood for the parameters $A_l = \{a_{ij,l}\}_{i,j=1}^{p}$, $l = 1, \ldots, k$ and μ can be expressed as

$$l(A_1, \ldots, A_k, \mu) = -\frac{p(T-k)}{2}\log(2\pi) - \frac{(T-k)}{2}\{\log[\det(\hat{\Omega})] + p\},$$

where $\hat{\Omega} = \sum_{t=k+1}^{T} \hat{\epsilon}_t \hat{\epsilon}_t'/(T-k)$ and $\hat{\epsilon}_t = \hat{\epsilon}_t(A_1, \ldots, A_k, \mu)$ is defined as solutions of (1). Finding the maximum likelihood estimates of $A_1, \ldots, A_k, \mu$ can therefore be done by minimizing $\log[\det(\hat{\Omega})]$. We will consider this problem, where in addition the parameters satisfy a set of restrictions that will be specified below.

The LRE hypotheses we shall study can be described by relations involving $k+1$ $p \times q$-dimensional matrices, $c_1, c_0, \ldots, c_{-k+1}$ and a q dimensional vector c where the matrices $d = -c_1 - c_0 - \cdots - c_{-k+1}$ and c_1 have rank q. Some of the entries may involve unknown structural or semi-structural parameters. The hypotheses take the following form where E_t denotes the conditional expectations given the observations up to time t, $X_{k+1}, \ldots, X_t$:

$$c_1' E_t[X_{t+1}] + c_0' X_t + c_{-1}' X_{t-1} + \cdots + c_{-k+1}' X_{t-k+1} + c = u_t. \tag{2}$$

The error terms u_t are a sequence of innovations in the VAR model, i.e. $E_t[u_{t+1}] = 0$.

If all $u_t = 0$ the LRE model is *exact*. The case where the innovation terms are non-zero, i.e. $E_t[u_t^2] > 0$, is denoted as the *inexact* LRE model.

Example 1. (a) The *New-Keynesian Phillips Curve, (NKPC)* model has the following form in the exact case

$$\pi_t = \gamma_f E_t[\pi_{t+1}] + \gamma_b \pi_{t-1} + \lambda s_t,$$

where π_t denotes log inflation and s_t is a proxy for the logarithm of marginal cost. With $X_t = (\pi_t, s_t, Z_t')'$, where the $p-2$ dimensional vector Z_t contains other relevant variables, this is exactly the case treated by Kurman (2007). Now $c_1 = (\gamma_f, 0, \ldots, 0)'$, $c_0 = (-1, \lambda, 0, \ldots, 0)'$, $c_{-1} = (\gamma_b, 0, \ldots, 0)'$ and $c = 0$.

(b) For the particular choice $\gamma_f = \delta$, $\gamma_b = 0$ and $\lambda = 1$ one gets a *present value model*. For $0 < \delta < 1$ the stable solution is $\pi_t = \sum_{i=0}^{\infty} \delta^i E_t[s_{t+i}]$. The case that $\pi_t = \sum_{i=1}^{\infty} \delta^i E_t[s_{t+i}]$ corresponds to $\pi_t = \delta E_t[\pi_{t+1} + s_{t+1}]$, where now $c_1 = (\delta, \delta, 0, \ldots, 0)'$, $c_0 = (-1, 0, 0, \ldots, 0)'$.

(c) *Alternative inflation model.* Sbrdone (2002) considered a bivariate model of the form $\Delta p_t = \delta E_t[\Delta p_{t+1}] - \alpha_1(p_t - ulc_t + \kappa)$, where p_t and ulc_t are the logarithm of price and unit labor cost, respectively. Then $X_t = (p_t, ulc_t)'$, $c_1 = (\delta, 0)'$, $c_0 = (-1-\delta-\alpha_1, \alpha_1)'$, $c_{-1} = (1, 0)'$ and $c = -\alpha_1\kappa$. ∎

Let $\mathbf{X}_t = (X_t', \ldots, X_{t-k+1}')'$. The companion form of (1) is defined as

$$\mathbf{A} = \begin{pmatrix} (A_1, A_2, \ldots, A_{k-1}) & A_k \\ I_{(k-1)p} & 0 \end{pmatrix}. \tag{3}$$

Then (1) can be expressed as $\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t$, where $\boldsymbol{\mu}_t = (\mu_t', 0, \ldots, 0)'$ and $\boldsymbol{\epsilon}_t = (\epsilon_t', 0, \ldots, 0)'$. For VAR-models $E_t[\mathbf{X}_{t+j}] = \mathbf{A}^j \mathbf{X}_t + \sum_{i=1}^{j} A^{i-1} \boldsymbol{\mu}_{t+j-i+1}$. Inserting in (2) and equating coefficients the restrictions defining the exact LRE model may be written as

$$\begin{aligned} & c_1' A_1 + c_0' = 0, c_1' A_2 + c_{-1}' = 0, \ldots, c_1' A_k + c_{-k+1}' = 0, \\ & c_1' \mu_{t+1} + c = 0, \quad t = k+1, \ldots, T. \end{aligned} \tag{4}$$

Similarly, one get for the inexact model by leading (2) one more lag and using iterated expectations that

$$\begin{aligned} & (c_1' A_1 + c_0') A_j + c_1' A_{j+1} + c_{-j}' = 0, \quad j = 1, \ldots, k-1 \\ & (c_1' A_1 + c_0') A_k = 0 \\ & (c_1' A_1 + c_0') \mu_{t+1} + c_1' \mu_{t+2} + c = 0, \quad t = k+1, \ldots, T. \end{aligned} \tag{5}$$

The conditions (4) and (5) are also necessary, so the following proposition holds.

Proposition 1. *For the VAR model (1) the following equivalences hold.*

(i) *The exact LRE model is equivalent to the restrictions (4).*
(ii) *That $\{u_t\}$ are innovations is equivalent to the restrictions (5).*

Proof. (i) The sufficiency is established above. On the other hand $E_t[X_{t+1}] = A_1 X_t + \cdots + A_k X_{t-k+1} + \mu_{t+1}$ from (1), so $c_1' E_t[X_{t+1}] + c_0' X_t + c_{-1}' X_{t-1} + \cdots + c_{-k+1}' X_{t-k+1} + c = (c_1' A_1 + c_0') X_t + (c_1' A_2 + c_{-1}') X_{t-1} + \cdots (c_1' A_k + c_{-k+1}') X_{t-k+1} + (c_1' \mu_{t+1} + c) = 0$ by (4).

(ii) The sufficiency is established above. To prove the necessity, we use the expression for $E_t[X_{t+1}]$ and the definition of u_t to write $u_t = (c_1' A_1 + c_0') X_t + (c_1' A_2 + c_{-1}') X_{t-1} + \cdots + (c_1' A_k + c_{-k+1}') X_{t-k+1} + (c_1' \mu_{t+1} + c)$. Hence, using the expression for $E_t[X_{t+1}]$ once more, $E_t[u_{t+1}] = [(c_1' A_1 + c_0') A_1 + (c_1' A_2 + c_{-1}')] X_t + [(c_1' A_1 + c_0') A_2 + (c_1' A_3 + c_{-2}')] X_{t-1} + \cdots + [(c_1' A_1 + c_0') A_{k-1} + (c_1' A_k + c_{-k+1}')] X_{t-k+2} + [(c_1' A_1 + c_0') A_k] X_{t-k+1} + [(c_1' A_1 + c_0') \mu_{t+1} + (c_1' \mu_{t+2} + c)]$. By (5) $E_t[u_{t+1}] = 0$. ∎