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Notes on supermodularity and increasing differences in expected utility

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HIGHLIGHTS

- I provide axiomatic foundations for supermodularity of Bernoulli utility functions.
- Axioms for quasi-supermodularity and supermodularity are compared.
- For parametric utilities, I provide axioms for increasing differences.
- Axioms for increasing differences and single crossing are compared.

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ABSTRACT

Many choice-theoretic and game-theoretic applications in Economics invoke some form of supermodularity or increasing differences for objective functions defined on lattices. These notes provide axiomatic foundations for these properties on expected-utility representations of preferences over lotteries.

1. Introduction

These notes revisit the axiomatic foundations of the properties of supermodularity and of increasing differences for expected-utility representations of preferences defined over lotteries.[1] While the first of these properties relates to a single preference relation and the second one involves a family of preferences, mathematically, they are closely related: a supermodular function on a product lattice has increasing differences.

2. Supermodular expected utility

2.1. Lattices and supermodularity

Let $(X, \geq_X)$ be lattice, and denote the "join" and "meet" operations by $\vee$ and $\wedge$, respectively. For any $f : X \to \mathbb{R}$, if $f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$ for any $x, x' \in X$, then f is supermodular. Following Li Calzi (1990) and Milgrom and Shannon (1994), f is quasi-supermodular if, for any $x, x' \in X$, we have that $f(x) \geq f(x \wedge x')$ implies $f(x \vee x') \geq f(x')$, with the corresponding implication for strict inequality. In words, for any $x, x' \in X$, if "meeting" x with x' "downgrades" x, then "joining" x and x' "upgrades" x', according to f. In other words, if the value of x under f is strictly higher than the value under f of $x \wedge x'$, then f cannot attain a higher value at x' than at $x \vee x'$.

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[1] Echenique and Chambers (2009) explores a related issue in the context of ordinal preferences on finite lattices.

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Supermodularity is a cardinal property, while quasi-supermodularity is an ordinal implication of supermodularity. Any non-decreasing function satisfies the weak part of quasi-supermodularity, and any strictly increasing function is quasi-supermodular.

2.2. Mixture spaces and Borel probability measures

Let $\mathcal{T}$ be a topology on X such that $(X, \mathcal{T})$ is a T1 space, that is, a space in which all singletons are closed sets. Denote by $\mathcal{B}(X)$ the Borel σ-field on X, and let $\Delta(X)$ denote the space of Borel probability measures on X. Finally, let $\Delta^0(X) \subseteq \Delta(X)$ be the subset of simple Borel probability measures on X, that is, the probability measures in $\Delta(X)$ that have finite support. In particular, for any $x \in X$, the point mass concentrated at x, δ_x, is an element of $\Delta^0(X)$.

A pair $(Z, *)$, where Z is a set and $*$ is an operation $* : [0, 1] \times Z \times Z \to Z$, is a mixture space (Fishburn, 1982) if the following hold.

- For all $z, z' \in Z$, $*(1, z, z') = z$.
- For all $z, z' \in Z$, and for all $\alpha \in [0, 1]$, $*(\alpha, z, z') = *(1 - \alpha, z', z)$.
- For all $z, z' \in Z$, and for all $\alpha, \beta \in [0, 1]$, $*(\alpha, *(\beta, z, z'), z') = *(\alpha\beta, z, z')$.

Both $\Delta(X)$ and $\Delta^0(X)$, coupled with the operation of taking convex combinations of probability measures, $*(\alpha, \mu, \mu') = \alpha\mu + (1 - \alpha)\mu'$, form mixture spaces. Henceforth, $*$ will denote this specific mixture operation.

Let $\mathcal{D} \subseteq \Delta(X)$ be a subset of probability measures that contains all simple probability measures and that forms a mixture space on its own; that is to say, $\Delta^0(X) \subseteq \mathcal{D}$ and $(\mathcal{D}, *)$ is a mixture space. A function $f : \mathcal{D} \to \mathbb{R}$ is linear in $*$ if, for all $\mu, \mu' \in \mathcal{D}$ and all $\alpha \in [0, 1]$, $f\left(*(\alpha, \mu, \mu')\right) = \alpha f(\mu) + (1 - \alpha) f(\mu')$.

2.3. Preferences over lotteries and supermodular expected utility

Let $\succsim \subseteq \mathcal{D} \times \mathcal{D}$ be a complete preorder on $\mathcal{D}$. Since $\mathcal{D}$ contains all point masses, $\succsim$ induces a complete preorder on X, $\succsim_X$, given by $x' \succsim_X x$ if $\delta_{x'} \succsim \delta_x$ for any $x, x' \in X$. The asymmetric and symmetric parts of $\succsim_X$, denoted by $\succ_X$ and $\sim_X$, respectively, are the relations induced by the asymmetric and symmetric parts of $\succsim$.

In choice-theoretic and game-theoretic applications, supermodularity is imposed on Bernoulli utility functions; these represent $\succsim_X$. However, the primitive preference is that over lotteries, namely $\succsim$. Hence, the relevant link is the link between supermodularity of representations of $\succsim_X$ and properties of $\succsim$.

In the Mixture Space Theorem (Herstein and Milnor, 1953), the following three axioms are imposed on $\succsim$.

(a) $\succsim$ is a complete preorder.
(b) For all $\mu, \mu', \mu'' \in \mathcal{D}$, and for all $\alpha \in (0, 1)$, $\mu' \succ \mu$ implies $*\left(\alpha, \mu', \mu''\right) \succ *\left(\alpha, \mu, \mu''\right)$.
(c) For all $\mu, \mu', \mu'' \in \mathcal{D}$ such that $\mu \succ \mu' \succ \mu''$, there exists some $\alpha, \beta \in (0, 1)$ such that $*\left(\alpha, \mu, \mu''\right) \succ \mu' \succ *\left(\beta, \mu, \mu''\right)$.

Axiom (a) is a necessary assumption for $\succsim$ to admit a numerical representation. Axiom (b) is an independence assumption, stating that the presence of a third lottery μ'' does not change the ranking of μ, μ' when mixed with "equal weight". Finally, Axiom (c) is a continuity or Archimedean axiom. Following Kreps (2013), this last axiom rules out the existence of "supergood" or "superbad" lotteries: no matter how high μ is ranked by the agent, for some mixture, μ' is still strictly preferred to this mixture of μ and μ''. Similarly for μ'': no matter how low it is ranked, μ' is still strictly worse than some mixture of μ and μ''.

The Mixture Space Theorem states that a binary relation $\succsim$ on $\mathcal{D}$ satisfies these three axioms if and only if there exists a real-valued function u on $\mathcal{D}$ representing $\succsim$ that is linear in $*$ and unique up to positive affine transformations. If $\mathcal{D} = \Delta^0(X)$, the von Neumann and Morgenstern Theorem (von Neumann and Morgenstern, 1953) establishes the existence of a real-valued function U on X that is also unique up to positive affine transformations and such that $u(\mu) = \int U d\mu$. The result extends to the case $\mathcal{D} = \Delta(X)$ if there exists a metric d on X such that (X, d) is separable, and if $\succsim$ is continuous in the topology of weak convergence.

The function U in the von Neumann and Morgenstern Theorem represents $\succsim_X$, as $U(x) = u(\delta_x)$. Hence, the problem is to establish a link between properties of $\succsim$ and supermodularity of U. The obvious link is given by the following axiom, Axiom (S).

Axiom (S). For all $x, x \in X$, $*\left(\frac{1}{2}, \delta_{x \wedge x'}, \delta_{x \vee x'}\right) \succsim *\left(\frac{1}{2}, \delta_x, \delta_{x'}\right)$.

Axiom (S) states that, for any two outcomes, the 50–50 mixture between the "highest" and the "lowest" of the two (under $\geq_X$) is weakly preferred to the 50–50 mixture between the outcomes. If we think of X as the product of two lattices, the axiom can be read as saying that a 50–50 mixture between "all-high" or "all-low" coordinates is weakly preferred to a 50–50 lottery between elements that feature both high and low coordinates. Thus, it can be read as an axiom about "complementarity across dimensions".

Theorem 1. *Let $(X, \geq_X)$ be a lattice, and let $\succsim$ be a binary relation on $\Delta^0(X)$. Then, $\succsim$ satisfies Axioms* (a), (b), (c), *and* (S) *if and only if there exists a supermodular real-valued function $U : X \to \mathbb{R}$ such that $u : \Delta^0(X) \to \mathbb{R}$ given by $u(\mu) = \sum_{x \in \text{supp}(\mu)} U(x)\mu(\{x\})$ represents $\succsim$. Moreover, U is unique up to positive affine transformations.*

The 50–50 mixture specified by Axiom (S) is crucial in the proof of Theorem 1. For other mixtures, quasi-supermodularity follows instead. Consider the following weaker version of Axiom (S), Axiom (qS).

Axiom (qS). For all $x, x \in X$, there exists some $\alpha \in (0, 1)$ such that $*\left(\alpha, \delta_{x \wedge x'}, \delta_{x \vee x'}\right) \succsim *\left(\alpha, \delta_x, \delta_{x'}\right)$.

Axiom (qS) states that, for any two outcomes, there exists a (strict) mixture between the "highest" and the "lowest" of the two that is weakly preferred to the same mixture between the outcomes themselves. However, this mixture may be different that $1/2$, and it may depend on the choice of $x, x' \in X$.[2]

Theorem 2. *Let $(X, \geq_X)$ be a lattice, and let $\succsim$ be a binary relation on $\Delta^0(X)$. Then, $\succsim$ satisfies Axioms* (a), (b), (c), *and* (S)*if and only if there exists a quasi-supermodular real-valued function $U : X \to \mathbb{R}$ such that $u : \Delta^0(X) \to \mathbb{R}$ given by $u(\mu) = \sum_{x \in \text{supp}(\mu)} U(x)\mu(\{x\})$ represents $\succsim$. Moreover, U is unique up to positive affine transformations.*

3. Increasing differences in expected utility

In this section, $(X, \geq_X)$ is a poset (not necessarily a lattice), and $\mathcal{R}_\Theta := \left\{\succsim^\theta : \theta \in \Theta\right\}$ is an indexed family of complete preorders on $\mathcal{D} \subseteq \Delta(X)$. The index set Θ is also endowed with a partial order, denoted by $\geq_\Theta$. Following Milgrom and Shannon (1994), a function $F : X \times \Theta \to \mathbb{R}$ satisfies the single-crossing property if, for each $x, x' \in X$ and each $\theta, \theta' \in \Theta$ such that $x' >_X x$ and $\theta' >_\Theta \theta$, $F(x', \theta) \geq F(x, \theta)$ implies $F(x', \theta') \geq F(x, \theta')$, and $F(x', \theta) > F(x, \theta)$ implies $F(x', \theta') > F(x, \theta')$; if we have $F(x', \theta') - F(x, \theta') \geq F(x', \theta) - F(x, \theta)$, then F has increasing differences.

[2] If the mixture in Axiom (qS) is uniform across x, then representations will satisfy the following property, weaker than supermodularity but stronger than quasi-supermodularity. A function $f : X \to \mathbb{R}$ defined on a lattice $(X, \geq_X)$ is *α-supermodular* if there exists some $\alpha \in [0, 1]$ such that, for all $x, x' \in X$, $\alpha f(x \wedge x') + (1 - \alpha) f(x \vee x') \geq \max\{\alpha f(x) + (1 - \alpha) f(x'), \alpha f(x') + (1 - \alpha) f(x)\}$.