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On the characteristic function for asymmetric Student t distributions

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HIGHLIGHTS

- Arguments are given for asymmetric Student t distributions to be the most versatile models for financial returns.
- Closed-form expressions are derived for the characteristic function of asymmetric Student t distributions.
- The use of characteristic functions in the econometric literature discussed.

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ABSTRACT

Following up on the work of Nadarajah and Teimouri [Nadarajah, S., Teimouri, M., 2012. On the characteristic function for asymmetric exponential power distributions. Econometric Reviews 31, 475–481], we derive here, for the first time, explicit closed-form expressions for the characteristic function of the asymmetric Student t distribution. The expressions involve hypergeometric and Bessel type functions.

1. Introduction

A random variable X is said to have the Student t distribution with degrees of freedom ν ($\nu > 0$) if its probability density function (PDF) is given by

$$f(x) = K(\nu)\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \tag{1.1}$$

for $-\infty < x < \infty$, where $K(\nu) = \Gamma\left((\nu+1)/2\right) / \left\{\sqrt{\pi\nu}\,\Gamma\left(\nu/2\right)\right\}$.

The Student t distribution has been the most popular model for financial returns. However, its major weakness is its inability to model asymmetric data. In recent years, the econometric literature has seen an explosion of activity to construct asymmetric generalizations of (1.1). We mention Theodossiou (1998), Bauwens and Laurent (2005), and Aas and Haff (2006). There are also numerous papers giving applications of these asymmetric generalizations to problems in economics and finance. The list is too exhaustive to mention here. We refer readers to the cited papers for some examples of applications.

The most general form of the Student t distribution developed to date is that due to Zhu and Galbraith (2010). Its PDF takes the form

$$f(x) = \begin{cases} \dfrac{\alpha}{\alpha^*}K\left(\nu_1\right)\left\{1+\dfrac{1}{\nu_1}\left[\dfrac{x}{2\alpha^*}\right]^2\right\}^{-\frac{\nu_1+1}{2}}, & \text{if } x \le 0, \\ \dfrac{1-\alpha}{1-\alpha^*}K\left(\nu_2\right)\left\{1+\dfrac{1}{\nu_2}\left[\dfrac{x}{2\left(1-\alpha^*\right)}\right]^2\right\}^{-\frac{\nu_2+1}{2}}, & \text{if } x > 0 \end{cases} \tag{1.2}$$

for $-\infty < x < \infty$, $0 < \alpha < 1$, $\nu_1 > 0$, and $\nu_2 > 0$, where $\alpha^* = \alpha K\left(\nu_1\right) / \left\{\alpha K\left(\nu_1\right) + (1-\alpha)K\left(\nu_2\right)\right\}$. Here, ν_1 and ν_2 are shape parameters. The distribution given by (1.2) is referred to as the asymmetric Student t distribution.

Many known generalizations of the Student t distribution to date are particular cases of (1.2). The generalized t distributions

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due to Hansen (1994) and Fernandez and Steel (1998) are particular cases of (1.2) for $\nu_1 = \nu_2$. The Student t distribution in (1.1) is the particular case of (1.2) for $\nu_1 = \nu_2$ and $\alpha = 1/2$. The generalized t distribution due to Aas and Haff (2006) exhibits the same tail behavior as (1.2) as $\nu_2 \to \infty$.

Zhu and Galbraith (2010) and the follow-up paper Zhu and Galbraith (2011) have derived various mathematical properties of (1.2). However, to the best of our knowledge, closed-form expressions for the characteristic function (CHF) have not been known for (1.2). The CHF is a fundamental tool in probability and statistics. CHFs can also be used as part of procedures for fitting distributions to samples of data.

Furthermore, the asymmetric Student t distribution in (1.2) and the asymmetric exponential power distribution due to Zhu and Zinde-Walsh (2009) have proved to be good competitors for modeling of financial returns. By means of GARCH modeling of the returns of the prices of Brent crude oil, west Texas intermediate crude oil, cocoa beans, gold, silver, and wheat, Nadarajah et al. (submitted for publication) have shown that both these distributions perform better than all others implemented in the R (R Development Core Team, 2013) contributed package fGarch (Wuertz and Chalabi, 2013).

Nadarajah and Teimouri (2012) have derived explicit closed-form expressions for the CHF for the asymmetric exponential power distribution. The aim of this short note is to derive explicit closed-form expressions for the CHF associated with (1.2). These expressions can be used to deduce closed-form CHFs for the various particular cases of (1.2).

Applications of CHFs are extensive in the econometric literature. As stated in Nadarajah and Teimouri (2012), some recent examples of their use include: "pricing and spanning opportunities for derivative-security valuation (Bakshi and Madan, 2000); estimation of continuous time stochastic models (Chacko and Viceira, 2003); calculation of power functions for unit root tests (Juhl and Xiao, 2003); recursions for compound phase distributions (Eisele, 2006); testing for conditional symmetry (Su, 2006); testing for conditional independence (Su and White, 2007); and, testing for multifactor continuous-time Markov models (Chen and Hong, 2010). The mentioned models and the mentioned tests are popular in economics and finance".

Some other recent applications of CHFs not stated in Nadarajah and Teimouri (2012) are: average rate claims with emphasis on catastrophe loss options (Bakshi and Madan, 2002); international cooperation, coalitions stability, and free riding in a game of pollution control (Breton et al., 2006); oil price dynamics (Askari and Krichene, 2008); in-sample and out-of-sample specification analysis of spot rate models (Cai and Swanson, 2011); and evaluation of European compound option prices (Griebsch, 2013).

In Section 2, we derive explicit closed-form expressions for the CHF for (1.2). Some of these expressions involve the hypergeometric functions

$$
{}_0F_1\left(;a;x\right) = \sum_{k=0}^{\infty} \frac{1}{(a)_k} \frac{x^k}{k!}
$$

and

$$
{}_1F_2\left(a;b,c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\,(c)_k} \frac{x^k}{k!},
$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. Hypergeometric functions are included as in-built functions in most popular mathematical software packages, including *Maple*, Matlab, and *Mathematica*.

2. Main results

Theorem 2.1 derives an explicit closed-form expression for the CHF for (1.2) in terms of the hypergeometric functions mentioned in Section 1. Theorem 2.2 derives an equivalent expression for the CHF in terms of the modified Bessel function of the first kind defined by

$$
I_\nu\left(x\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)k!} \left(\frac{x}{2}\right)^{2k+\nu}.
$$

Theorem 2.3 derives an equivalent expression for the CHF in terms of the Bessel function of the first kind defined by

$$
J_\nu\left(x\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)k!} \left(\frac{x}{2}\right)^{2k+\nu}.
$$

Theorem 2.4 derives an equivalent expression for the CHF in terms of the Struve L function defined by

$$
J_\nu\left(x\right) = \left(\frac{x}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k}.
$$

Theorem 2.5 derives an equivalent expression for the CHF in terms of the Struve H function defined by

$$
H_\nu\left(x\right) = \left(\frac{x}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k}.
$$

The modified Bessel function of the first kind, the Bessel function of the first kind, the Struve L function, and the Struve H function are at least as popular as the hypergeometric functions. In-built routines for them are at least as widely available as those for the hypergeometric functions.

Theorem 2.1. *Let X be a random variable having the PDF (1.2). Let $\phi_X(t) = E[\exp(\mathrm{i}tX)]$ denote the* CHF *of X, where $\mathrm{i} = \sqrt{-1}$. Then $\phi_X(t)$ can be expressed as*

$$
\begin{aligned}
\phi_X(t) = &-\left(-\mathrm{i}t\right)^{\nu_1} K\left(\nu_1\right) \nu_1^{\frac{\nu_1+3}{2}} 2^{\nu_1+1}\alpha\left(\alpha^*\right)^{\nu_1} \\
&\times {}_0F_1\left(;\frac{\nu_1}{2}+1; t^2\nu_1\left(\alpha^*\right)^2\right) + 2\alpha\, {}_0F_1\left(;1-\frac{\nu_1}{2}; t^2\nu_1\left(\alpha^*\right)^2\right) \\
&+\frac{\nu_1 \mathrm{i}t}{\nu_1-1} K\left(\nu_1\right) 4\alpha\alpha^* {}_1F_2\left(1;\frac{3}{2},\frac{3-\nu_1}{2}+1; t^2\nu_1\left(\alpha^*\right)^2\right) \\
&-\left(\mathrm{i}t\right)^{\nu_2} K\left(\nu_2\right) \nu_2^{\frac{\nu_2+3}{2}} 2^{\nu_2+1}(1-\alpha)\left(1-\alpha^*\right)^{\nu_2} \\
&\times {}_0F_1\left(;\frac{\nu_2}{2}+1; t^2\nu_2\left(1-\alpha^*\right)^2\right) \\
&+2(1-\alpha)\, {}_0F_1\left(;1-\frac{\nu_2}{2}; t^2\nu_2\left(1-\alpha^*\right)^2\right) \\
&-\frac{\nu_2 \mathrm{i}t}{\nu_2-1} K\left(\nu_2\right) 4(1-\alpha)\left(1-\alpha^*\right) \\
&\times {}_1F_2\left(1;\frac{3}{2},\frac{3-\nu_2}{2}+1; t^2\nu_2\left(1-\alpha^*\right)^2\right)
\end{aligned}
$$

for $0 < \alpha < 1$, $\nu_1 > 0$ and $\nu_2 > 0$.

Proof. Note that we can write

$$
\begin{aligned}
\phi_X(t) &= \int_{-\infty}^{\infty} \exp\left(\mathrm{i}tx\right) f(x)dx \\
&= \int_{-\infty}^{0} \exp\left(\mathrm{i}tx\right) f(x)dx + \int_{0}^{\infty} \exp\left(\mathrm{i}tx\right) f(x)dx \\
&= \frac{\alpha}{\alpha^*} K\left(\nu_1\right) \int_{-\infty}^{0} \exp\left(\mathrm{i}tx\right) \left\{1+\frac{1}{\nu_1}\left[\frac{x}{2\alpha^*}\right]^2\right\}^{-\frac{\nu_1+1}{2}} dx
\end{aligned}
$$