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Stable lexicographic rules for shortest path games

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HIGHLIGHTS

- We study the class of shortest path games.
- We propose new cost sharing rules satisfying core selection.
- These rules allocate shares according to some lexicographic preference relation.
- A computational procedure is provided.

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ABSTRACT

For the class of shortest path games, we propose a family of new cost sharing rules satisfying core selection. These rules allocate shares according to some lexicographic preference relation. A computational procedure is provided. Our results relate to those of Tijs et al. (2011).

1. Introduction

In this paper we consider shortest path problems, which arise in networks where the cost of shipping demand units through each arc is linear. This framework allows to model public transportation systems (connecting several urban areas) among other applications.

Each agent has to ship their demand from the source to their location, using the cost-minimizing route. In doing so, they might have to use the locations of other agents. There are two important issues in this context: (i) computing the shortest path network and (ii) designing sensible cost allocations for the resulting cooperative game. The issue (i) has been solved by Dijkstra (1959), who proposed an algorithm allowing to compute the shortest path to any location in a network. In this paper, we focus on (ii).

Shortest path problems are part of the larger family of network (and source-connection) models — see Sharkey (1995) for a review. It is known from Rosenthal (2013) that: (a) the core of a shortest path game is nonempty and (b) the Shapley value may produce an allocation that is not stable. A trivial core allocation is the one that charges the cost of their shortest path to the demander of each unit. This method does not remunerate agents whose cooperation helps the demanders reduce their connection cost. Surprisingly, the literature has not proposed stable cost sharing methods that are more sensible than the aforementioned rule. The objective of the present paper is to address this deficiency. Our approach is in the vein of cost sharing problems with technological cooperation introduced in Bahel and Trudeau (2013).

We propose a family of methods that assign cost shares according to some lexicographic ordering. These methods are shown to produce an extreme allocation in the core of every shortest path problem. Our results relate to those of Tijs et al. (2011), who showed the existence of these lexicographic extrema in the core of balanced games. Our algorithm allows to compute these extrema

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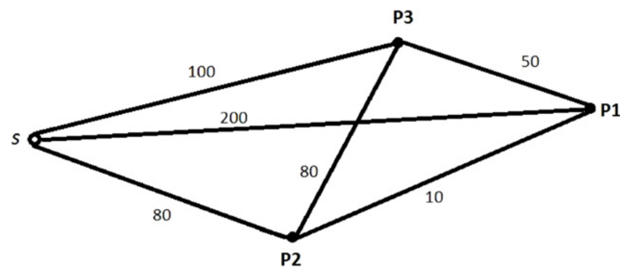

Fig. 1. A three-agent SPP.

for shortest path games. The average of these lexicographic rules shares many properties with the nucleolus (core selection, symmetry, continuity, ...) and it has the advantage (over the nucleolus) of being computable via our procedure.[1]

2. Preliminaries

Let us consider a fixed point s from which agents (residing at various locations) need to ship their respective demands of some homogeneous goods — s is called *the source*. A *Shortest Path Problem* (SPP) is a tuple (N, c, x), where (i) N is the set of agents (or vertices) that need to connect to the source s; (ii) $c = \{c(i, j)|i, j \in N \cup \{s\}, i \neq j\}$ is a collection (of nonnegative numbers) giving the unit cost of shipping demands through every edge (i, j) s.t. $i \neq j$; (iii) $x \in \mathbb{N}^N$ is the demand profile: each agent i has $x_i \in \mathbb{N}$ units of demand to ship from the source to her location. Note that we do not view the source s as a player. Also observe that the unit costs $c(i, j)$ need not be symmetric: we may have $c(i, j) \neq c(j, i)$ for some i and j. If instead $c(i, j) = c(j, i)$ for any distinct $i, j \in N \cup \{s\}$, we will say that the SPP has *symmetric arcs*.

For the rest of this section, we consider a fixed set of agents N, and a fixed cost structure c. Only the demand profile x may vary from one problem to the other.

Definition 1. Let $i \in N$. We call *path* (of length K) to i any sequence $p \equiv (p_k)_{k=0,...,K}$ such that: (i) $p_k \in N \cup \{s\}$, for $k = 0, 1, \ldots, K$; (ii) $p_0 = s$ and $p_K = i$; (iii) $p_k \neq p_{k'}$ for any distinct k, k'.

Note that all paths p originate from the source s and cross any location p_k only once. Thus, the length of each path and the number of paths to any given $i \in N$ are both finite. We denote by $\mathcal{P}(i)$ the set containing all paths to i. For any path p of length K, let $[p]$ refer to the set of players in the range of p, that is: $[p] \equiv \{i \in N|\ p_k = i \text{ for some } k = 1, \ldots, K\}$. For any subset $M \subsetneq N$ and any path p (of length K) such that $M \subsetneq [p]$, we will write $p \setminus M$ to refer to the unique path (of length $K - |M|$) where the agents of M have been deleted and the remaining agents (of $[p]$) appear in the same order as in p. To ease on notation, we will often write i instead of $\{i\}$ and $p \setminus i$ instead of $p \setminus \{i\}$, for any $i \in [p]$. Finally, for any $M \subseteq N$ we will write $\Pi(M)$ to refer to the set containing all permutations of M.

Given $P = (N, c, x)$, one can extend the cost function c to paths as follows: for any path p (of length K) to i,

$$c(p) = \sum_{k=1}^{K} c(p_{k-1}, p_k).$$

In words, $c(p)$ stands for the cost of shipping one unit from the source to agent i via the path p. For any $i \in N$, we call *shortest path to* i any path $p^* \in \mathcal{P}(i)$ that solves the problem $\min_{p\in\mathcal{P}(i)} c(p)$. Note that there exists a shortest path to any $i \in N$ — since the set $\mathcal{P}(i)$ is nonempty and finite; but it may not be unique.

Example 1. Consider the SPP (with symmetric arcs) given by $P = (N, c, x)$, where $N = \{1, 2, 3\}$, $x = (2, 0, 1)$ and the cost structure is depicted by Fig. 1. For example, we have $c(s, 1) = 200$, $c(3, 1) = c(1, 3) = 50$ and $c(2, 1) = c(1, 2) = 10$.

One can see that there are 5 paths to agent 1, $(s, 1)$, $(s, 2, 1)$, $(s, 3, 1)$, $(s, 2, 3, 1)$, $(s, 3, 2, 1)$; and the shortest path to 1 is $(s, 2, 1)$, with cost $c(s, 2, 1) = 80 + 10 = 90$. For agents 2 and 3, the costs of the respective shortest paths are $c(s, 2) = 80$ and $c(s, 3) = 100$.

For any vector $y \in \mathbb{R}^N$ and any subset $S \subseteq N$, let $y(S) \equiv \sum_{i\in S} y_i$. It is not difficult to see that there is a natural way to formulate the cooperative game (with transferable cost) associated with P.

Define the cost of any nonempty coalition $S \subseteq N$ as follows:

$$C_P(S) \equiv \min \left\{ \sum_{j\in S} x_j c(p^j) \mid p^j \in \mathcal{P}(j) \text{ and } [p^j] \subseteq S, \forall j \in S \right\}. \quad (1)$$

Eq. (1) gives the lowest possible cost of shipping (from the source) the respective demands of the members of S when using only the connections available in S. Note in particular that $C_P(S) = 0$ whenever $x(S) = 0$ (there is no demand to ship). We also adopt the usual convention that $C_P(\emptyset) = 0$. As an illustration, for the problem P depicted in Example 1, note that we have $C_P(N) = 2 \times c(s, 2, 1) + 0 \times c(s, 2) + 1 \times c(s, 3) = 180 + 100 = 280$.

Definition 2. Given a shortest path problem $P = (N, c, x)$, we have the following.

(i) An *allocation* is a profile of cost shares, $y \in \mathbb{R}^N$, such that $y(N) = C_P(N)$. Let $\mathcal{A}(P)$ be the set containing all cost allocations.

(ii) The *core* of P is the set

$$Core(P) \equiv \{y \in \mathcal{A}(P)|y(S) \leq C_P(S), \text{ for any } S \text{ s.t. } \emptyset \neq S \subset N\}.$$

An allocation y will be called *stable* if $y \in Core(P)$.

Thus, a cost allocation is a way of "splitting" the cost $C_P(S)$ of the coalition S between its members. Note that we allow for negative cost shares, which are desirable if we have (for instance) agents who demand zero while providing the others with a cheap connection to the source. Definition 2-(ii) is the standard notion of stability: no coalition S should jointly pay more than its stand-alone cost $C_P(S)$.

3. Analysis

In the remainder of the paper, unless otherwise specified, we consider an arbitrary but fixed SPP given by $P = (N, c, x)$, where $0^N \neq x \in \mathbb{N}^N$.

3.1. Decomposition

In an SPP, shipping one unit to a given agent does not affect the cost of shipping the next unit (to any agent). Using this observation, we first study "elementary" SPP, which have the property that only one agent has a (unitary) demand.

For every $j \in N$, denote by $e^j \in \mathbb{R}^N$ the vector characterized by $e^j_j = 1$ and $e^j_i = 0$, if $i \in N \setminus j$. Let $A, B \subset \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. We use the following conventions: $A + B \equiv \{a + b|a \in A \text{ and } b \in B\}$; $\alpha \cdot A \equiv \{\alpha a|a \in A\}$. This notation allows to write the result hereafter.

Lemma 1. *Given the problem $P = (N, c, x)$, we have*

$$\sum_{j\in N} x_j \cdot Core(P^j) \subseteq Core(P),$$

where $P^j \equiv (N, c, e^j)$.

The proof is omitted. It trivially follows from Eq. (1) and Definition 2.

[1] The two solution concepts are different, as pointed out by Tijs et al. (2011).