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Effect of the order of fractional integration on impulse responses

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HIGHLIGHTS

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ABSTRACT

For a fractional time series model integrated of order d we derive two results. First, it is obtained how a change in d affects the coefficients of the integration filter. For long memory ($d > 0$), the effect is always positive; in the case of anti-persistence ($d < 0$) the effect may be positive or negative depending on d. Second, those results are extended to the sequence of autocorrelations for fractionally integrated noise.

1. Introduction

The persistence in univariate time series is often measured by means of the cumulated impulse response [IR] function or related measures, see e.g. Cogley and Sargent (2005) or Paya et al. (2007), and Campbell and Mankiw (1987) or Cochrane (1988) for earlier examples. If, however, the data is from a process that is (fractionally) integrated of order $d \neq 0$, $I(d)$, then Hauser et al. (1999) pointed out that the cumulated IR function is meaningless: it amounts to zero if $d < 0$ and is unbounded if $d > 0$. The latter situation is usually characterized as "long memory", while the first one has been called "anti-persistence". See Baillie (1996) or Brockwell and Davis (1991, Section 13.2) for expositions introducing into fractional integration [FI]. Under FI, persistence is hence not reflected by the cumulated IR coefficients but by the memory parameter d. Therefore, it is of applied interest how a change in d affects the individual IR coefficients. For shocks j periods in the past and j being large, the IR coefficients die out hyperbolically with rate j^{d-1}, see Hassler and Kokoszka (2010) and Hassler (2012) for minimal assumptions under $d > 0$ and $d < 0$, respectively. In this paper, we are interested on the effect of d on the IR coefficients for shocks being past by only 1, 2 or a small number j of periods.

We will look more closely at the situation of so-called fractionally integrated noise [FN], where differencing the data with a difference filter of order d results in a white noise sequence free of serial correlation. Such a parsimonious model could not be rejected when characterizing time series like inflation rates (see Hassler and Wolters, 1995) or realized volatility (see Christensen and Nielsen, 2007), and also opinion poll series (see Byers et al., 1997, 2007; Mayoral et al., 2003).

The rest of this paper is organized as follows. The next section summarizes relevant results for fractionally integrated processes. Section 3 contains our new results with empirically relevant interpretations. Proofs are provided in Section 4. The last section concludes.

2. Fractional integration

Let $\{y_t\}$, $t \in \mathbb{Z}$, be a stationary, fractionally integrated process, $I(d)$ with $d \neq 0$,

$$y_t = (1 - L)^{-d} x_t, \quad -1 < d < 0.5,$$

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where $(1-L)^{-d}$ is given by the usual binomial expansion in terms of the lag operator L. The process $\{x_t\}$ is $I(0)$ in that its Wold representation is absolutely summable, and the sequence of impulse response coefficients $\{b_k\}$ does not sum up to zero:

$$x_t = B(L)\varepsilon_t = \sum_{k=0}^{\infty} b_k \varepsilon_{t-k}, \quad \sum_{k=0}^{\infty} |b_k| < \infty, \quad \sum_{k=0}^{\infty} b_k \neq 0,$$

where $\{\varepsilon_t\}$ forms a stationary martingale difference series with variance σ^2. The special case where $B(L) = 1$, $x_t = \varepsilon_t$, is called fractionally integrated noise. For

$$(1-L)^{-d} = \sum_{j=0}^{\infty} \psi_j(d) L^j, \quad \psi_0(d) = 1,$$

we obtain with the Gamma function $\Gamma(\cdot)$

$$\psi_j(d) = \frac{j-1+d}{j}\psi_{j-1}(d) = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \quad j \geq 1, \tag{1}$$

such that

$$\psi_j(d) \sim \widetilde{\psi}_j(d), \quad j \to \infty, \text{ with } \widetilde{\psi}_j(d) = \frac{1}{\Gamma(d)} j^{d-1}; \tag{2}$$

see e.g. Granger and Joyeux (1980), Hosking (1981) or Brockwell and Davis (1991, Section 13.2). In the case of FN, $\{\psi_j(d)\}$ is the IR sequence of $y_t = (1-L)^{-d}\varepsilon_t$. For $B(L) \neq 1$, the IR coefficients of

$$y_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$$

are given by convolution of $(1-L)^{-d}$ and $B(L)$:

$$c_j(d) = \sum_{k=0}^{j} \psi_k(d) b_{j-k}. \tag{3}$$

We maintain the additional assumption

$$b_k = o(k^{d-1}). \tag{4}$$

In the case of long memory ($d > 0$), Hassler and Kokoszka (2010) proved that (4) is necessary and sufficient for the hyperbolic decay in (2) to carry over to the IR coefficients:

$$c_j(d) \sim \widetilde{c}_j(d), \quad j \to \infty, \text{ with } \widetilde{c}_j(d) = \frac{\sum_{k=0}^{\infty} b_k}{\Gamma(d)} j^{d-1}. \tag{5}$$

Under a slightly stronger sufficient condition on $\{b_k\}$, Hassler (2012) established (5) for the anti-persistent case ($d < 0$), too.

Now, we turn to the effect of changes in the memory parameter d on the impulse response coefficients. Under long memory, it is straightforward to see in the long run that $\widetilde{\psi}_j(d)$ and $\widetilde{c}_j(d)$ are increasing with growing d: The larger the memory parameter, the more persistent is the process. The same will be shown to hold true for $\psi_j(d)$ and $c_j(d)$ in the next section. In the case of short memory or anti-persistence ($d < 0$), matters are more delicate. On the one hand, we observe for large j from (2) and (5) that "The smaller the d (i.e. the more negative d), the faster the IR coefficients die out [...]" (Hassler, 2012, Remark E). On the other hand, for $d < 0$:

$$\sum_{j=0}^{\infty} \psi_j(d) = (1-z)^{-d}\Big|_{z=1} = 0.$$

Consequently, since $\psi_j'(d) < 0$ for large j, there must be a small j with opposite effect: $\psi_j'(d) > 0$. In the next section, we will spell out, how exactly $\psi_j(d)$ reacts to changes in the order of fractional integration.

In the case of FN ($x_t = \varepsilon_t$), which is of special interest, the autocorrelations $\rho_j(d)$ at lag j are known from Hosking (1981):

$$\rho_j(d) = \frac{\Gamma(j+d)\Gamma(1-d)}{\Gamma(j+1-d)\Gamma(d)}, \quad j = 1, 2, \ldots. \tag{6}$$

The hyperbolical decay of $\{c_j(d)\}$ translates into

$$\rho_j(d) \sim \widetilde{\rho}_j(d), \quad j \to \infty, \text{ with } \widetilde{\rho}_j(d) = \frac{\Gamma(1-d)}{\Gamma(d)} j^{2d-1}.$$

The influence of d on $\widetilde{\rho}_j(d)$ is obvious. In the next section, we discuss the influence for small j.

3. Results and discussion

The following proposition contains a neat expression for the derivative $\psi_j'(d) = \partial\psi_j(d)/\partial d$.

Proposition 1. *Let $\{\psi_j(d)\}$ be from (1) with $-1 < d < 0.5$, $d \neq 0$. Then*

$$\psi_j'(d) = \psi_j(d) D_j(d), \quad j = 1, 2, \ldots,$$

where

$$D_j(d) = \frac{1}{d} + \frac{1}{1+d} + \cdots + \frac{1}{j-1+d}.$$

Proof. See the next section.

Note that the sign of $\psi_j(d)$ equals that of d:

$$\psi_j(d) > 0 \iff d > 0.$$

Hence, we have the following remark for $d > 0$.

Remark 1. Under long memory it holds $D_j(d) > 0$, such that $\psi_j'(d) > 0$ for all j: As d grows, the sequence $\{\psi_j(d)\}$ is shifted upwards: The larger the d, the larger the IR of FN. The same holds true for general $I(d)$ processes with long memory if $\{b_k\}$ is a positive sequence, see (3).

In the case of anti-persistence the sign effect of a variation in d is more complicated and characterized in the following corollary.

Corollary 1. *Under $-1 < d < 0$ it holds $\psi_1'(d) > 0$ and*

$$\psi_j'(d) \begin{cases} <0 & \text{for } d < d_j \\ >0 & \text{for } d > d_j \end{cases}, \quad j = 2, 3, \ldots,$$

where d_j is the unique root of $D_j(d) = 0$ on $(-1, 0)$.

Proof. See next section.

Under anti-persistence, we learn for $j = 1$: the closer the d is to zero, the closer the $\psi_1(d)$ is to zero. Or the other way round: as d gets more negative, $|\psi_1(d)|$ increases. For $j \geq 2$, the sign of the derivative depends on d. For "very negative d", i.e. $d < d_j$, we have $\psi_j'(d) < 0$, and oppositely for d closer to 0. If $\{b_k\}$ is a known sequence, one may hence conclude the effect of d on $\psi_j(d)$. The following remark spells out the examples $j = 2$ and $j = 3$.

Remark 2. The unique roots of $D_j(d) = 0$ can easily be determined in practice under anti-persistence. For instance,

$$D_2(d) = \frac{2d+1}{d(1+d)} = 0 \quad \text{for } d = d_2 = -0.5.$$

Similarly, $D_3(d) = 0$ for $3d^2 + 6d + 2 = 0$, i.e. for $d = d_3 = \sqrt{1/3} - 1 \approx -0.423$.