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Iterative algorithm for non parametric estimation of the instrumental variables quantiles ☆

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HIGHLIGHTS

- Non parametric instrumental variables quantile estimation.
- Landweber iterative algorithm.
- Ill-posed Integral equation.
- Kernel smoothing.

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ABSTRACT

This paper proposes a simple algorithm for the numerical computation of the non parametric IV quantile estimation. This algorithm is based on the Landweber iterations for solving a nonlinear integral equation. The paper is illustrated by numerical simulations.

1. Introduction

The recent econometric literature has shown the relevance of the instrumental variables quantile function defined in the following way. Let $Y \in \mathbb{R}$, $Z \in \mathbb{R}^p$ and $W \in \mathbb{R}^q$ be three random vectors. We assume the existence of a function $\varphi : \mathbb{R}^p \times [0, 1] \to \mathbb{R}$ and of a random real number U such that:

$$\begin{cases} Y = \varphi(Z, U) \\ \varphi(Z, \cdot) \text{ is an increasing function} \\ U \sim \text{ uniform distribution in } [0, 1] \\ U \perp\!\!\!\perp W \ (U \text{ and } W \text{ are independent}). \end{cases}$$

This type of model has been studied by Chernozhukov and Hansen (2005), Chernozhukov et al. (2007), Horowitz and Lee (2007), Imbens and Newey (2009), Matzkin (2003), Gagliardini and Scaillet (2012) among others. Different identification conditions have been established and are essentially based on a conditional completeness condition:

$E(a(Z, U)|W, U) = 0$ *implies* $a(Z, U) = 0$ *for any square integrable function a* (see Chen et al. (forthcoming)).

As pointed out by Horowitz and Lee (2007) the function φ may be viewed as the solution of a nonlinear integral equation depending on the distribution of $X = (Y, Z, W)$ and a non parametric estimation of φ may be performed in two steps: first some characteristics of the distribution of X are estimated and second φ is estimated as a solution of the equation. This resolution is actually ill-posed and does not defined a consistent estimator. In order to solve this problem a regularization method is necessary (see Carrasco et al., 2006). The methodology of this approach has been presented in Horowitz and Lee (2007), Gagliardini and Scaillet (2012) and in a more general way by Dunker et al. (2014). Consistency and speed of convergence have been derived in these papers. However the numerical implementation is very often based on sieves approach which reduces to a parametric estimation.

The objective of this paper is to propose an efficient iterative method for estimating non parametrically φ which does not re-

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quire a sieve approximation of φ and which is based on kernel estimation of the distribution of X. Our approach is an application of Landweber iterations mechanism (see Engle et al., 1996 or Kaltenbacher et al., 2008). This paper is limited to the description of a practical and easy application of this method. We recall the general framework in Section 2; Section 3 presents the estimation procedure and Section 4 gives some simulation examples.

2. The IV quantile as a nonlinear inverse problem

The search for φ is essentially based on the independence condition between U and W and on the uniform distribution of U. These two assumptions imply:

$$P(U \leq u|W = w) = u \quad \forall u \in [0, 1], \tag{2.1}$$

or alternatively,

$$\int P(U \leq u, Z = z|W = w)dz = u. \tag{2.2}$$

Thus, using the monotonicity of φ, we obtain:

$$\int P(Y \leq \varphi(z, u), Z = z|W = w)dz = u. \tag{2.3}$$

Let us introduce the notation:

$$F(y, z|w) = P(Y \leq y, Z = z|W = w). \tag{2.4}$$

The object of interest φ is the solution of:

$$T(\varphi)(w, u) = \int F(\varphi(z, u), z|w)dz = u, \tag{2.5}$$

which defines a nonlinear integral equation problem.

More precisely T is an operator which transforms a function $\varphi \in L^2_{Z\times U}$ into a function in $L^2_{W\times U}$. The sets $L^2_{Z\times U}$ and $L^2_{W\times U}$ are the spaces of square integrable functions of (Z, U) and (W, U), with respect to the true distribution of Z given U and W given U. Under the uniformity assumption on U, the conditional densities are identical to the joint distribution of (Z, U) and (W, U). Under some regularity conditions, T is a Fréchet differentiable operator and its derivative at the true value φ verifies:

$$T'_{\varphi}(\tilde{\varphi}) = \int \tilde{\varphi}(z, u)f(\varphi(z, u), z|w)dz \tag{2.6}$$

where $\tilde{\varphi}$ is any element of $L^2_{Z\times U}$ and $f(y, z|w)$ is the density of Y and Z given $W = w$. This computation is obtained by computing the Gâteaux derivative (we replace φ by $\varphi + t\tilde{\varphi}$ and we compute the derivative with respect to t in $t = 0$) and by checking that this linear operator is actually the Fréchet derivative. The linear operator T'_{φ} is also an operator from $L^2_{Z\times U}$ into $L^2_{W\times U}$ and has an adjoint operator characterized by

$$\int T'_{\varphi}(\tilde{\varphi})\psi(w|u)f(w|u)dw = \int \varphi(z, u)T'^{*}_{\varphi}(\psi)f(z|u)dz. \tag{2.7}$$

The adjoint operator $T'^{*}_{\varphi}(\psi)$ maps $L^2_{W\times U}$ to $L^2_{Z\times U}$ and verifies:

$$T'^{*}_{\varphi}(\psi) = \int \psi(w, u)\frac{f(\varphi(z, u), z|w)f(w|u)}{f(z|u)}dw. \tag{2.8}$$

Using the independence condition between W and U this expression reduces to

$$T'^{*}_{\varphi}(\psi) = \int \psi(w, u)\frac{f(\varphi(z, u), z, w)}{f(z|u)}dw. \tag{2.9}$$

The Landweber algorithm to solve Eq. (2.5) is then the following: we assume that we may start with some $\varphi_0(z, u)$ not too far from φ and the recurrence equation is:

$$\varphi_{k+1}(z, u) = \varphi_k(z, u) + T'^{*}_{\varphi_k}(u - T(\varphi_k)) \tag{2.10}$$

or

$$\varphi_{k+1}(z, u) = \varphi_k(z, u) + \int \left\{u - \int F(\varphi_k(\zeta, u), \zeta|w)d\zeta\right\} \times \frac{f(\varphi_k(z, u), z, w)}{f(z, u)}dw. \tag{2.11}$$

This algorithm converges to a fixed point of Eq. (2.10) which is obviously a solution of (2.5) as in Kaltenbacher et al. (2008). If T is unknown and only estimated with an error term, the algorithm should be stopped at some stage k in order to regularize the ill-posed inverse problem. The choice of this stopping rule will be discussed in the next section.

3. An efficient estimation method

Let us assume that we observe an *i.i.d.* sample (y_i, z_i, w_i) of size n. We need first to estimate $F(y, z|w)$ and we adopt a kernel smoothing method:

$$\hat{F}(y, z|w) = \frac{\sum_{i=1}^{n} \mathbb{1}(y_i \leq y)\frac{1}{h_z}K\left(\frac{z-z_i}{h_z}\right)K\left(\frac{w-w_i}{h_w}\right)}{\sum_{i=1}^{n} K\left(\frac{w-w_i}{h_w}\right)} \tag{3.1}$$

where K is a univariate or multivariate kernel, h_w and h_z are bandwidths.

Then $T(\varphi)$ is estimated through:

$$\hat{T}(\varphi) = \int \hat{F}(\varphi(z, u), z|w)dz. \tag{3.2}$$

Using the approximation

$$\int \mathbb{1}(y_i \leq \varphi(z, u))\frac{1}{h_z}K\left(\frac{z - z_i}{h_z}\right)dz \simeq \mathbb{1}(y_i \leq \varphi(z_i, u)), \tag{3.3}$$

our estimator becomes:

$$\hat{T}(\varphi) = \frac{\sum_{i=1}^{n} \mathbb{1}(y_i \leq \varphi(z_i, u))K\left(\frac{w-w_i}{h_w}\right)}{\sum_{i=1}^{n} K\left(\frac{w-w_i}{h_w}\right)}.$$

The adjoint operator T'^{*}_{φ} is estimated by replacing $f(\varphi(z, u), z|w)$ and $f(z|w)$ by their kernel estimators:

$$\hat{f}(y, z, w) = \frac{1}{n}\sum \frac{1}{h_y}K\left(\frac{y - y_i}{h_y}\right) \times \frac{1}{h_z}K\left(\frac{z - z_i}{h_z}\right)\frac{1}{h_w}K\left(\frac{w - w_i}{h_w}\right) \tag{3.4}$$

and

$$\hat{f}(z|u) = \frac{1}{n}\sum \frac{1}{h_z}K\left(\frac{z - z_i}{h_z}\right)\frac{1}{h_u}K\left(\frac{u - u_i}{h_u}\right), \tag{3.5}$$

thanks to the knowledge of the marginal density of U (which is equal to 1 in the [0, 1] interval). In (3.5) u_i is replaced by the solution of $y_i = \varphi(z_i, u_i)$. Then we get

$$\hat{T}'^{*}_{\varphi}(\psi) = \frac{\sum \psi(w_i, u)\frac{1}{h_y}K\left(\frac{\varphi(z,u)-y_i}{h_y}\right)K\left(\frac{z-z_i}{h_z}\right)}{\sum K\left(\frac{z-z_i}{h_z}\right)\frac{1}{h_n}K\left(\frac{u-u_i}{h_u}\right)}$$