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Cumulative offer process is order-independent

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HIGHLIGHTS

- We study the standard matching-with-contracts model.
- Bilateral substitutability and irrelevance of rejected contracts are assumed.
- It is shown that the cumulative offer process is order-independent.
- This result does not necessarily hold without bilateral substitutability.

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ABSTRACT

This note shows that in the matching-with contracts model, the outcome of the cumulative offer process is order-independent if every hospital has a choice function that satisfies the bilateral substitutability condition and the irrelevance of rejected contracts condition.

1. Introduction

Since Hatfield and Milgrom (2005), the matching-with-contracts model has attracted growing attention by matching theorists, and a number of real-world applications have been proposed (Kominers and Sönmez, 2013; Sönmez, 2013; Sönmez and Switzer, 2013). In this literature, the *cumulative offer process* (henceforth, COP), which is an extension of Gale and Shapley's (1962) deferred acceptance algorithm, plays an important role: It can find a feasible and stable allocation as its output, under a condition called *bilateral substitutability*, which is not sufficient for the deferred acceptance algorithm to return a stable allocation.[1]

Despite its importance, however, the definition of the COP remains ambiguous. On the one hand, Hatfield and Kojima (2010) define the process so that only a single offer is made at each step. Yet, they do not fully specify the order in which agents make offers. Hence, strictly speaking, they give a definition of a class of algorithms rather than a unique algorithm. On the other hand, in Hatfield and Milgrom's (2005) original definition, multiple agents simultaneously make an offer at each step. Then, two natural questions would arise: First, does every process in the class defined by Hatfield and Kojima (2010) lead to an identical final outcome? Second, is the definition of Hatfield and Milgrom (2005) outcome-equivalent to that of Hatfield and Kojima (2010)? If the answers to these questions are negative, there could arise another question of in which order agents should make offers so as to achieve "better" outcomes in terms of efficiency, fairness, etc.

The purpose of the present note is to show that the COP is actually order-independent if every hospital has a choice function that satisfies the bilateral substitutability condition and the

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[1] For the relationships among substitutability conditions, see Hatfield and Kojima (2008, 2010).

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irrelevance of rejected contracts condition. That is, under these two conditions, the final outcome of the COP is independent of what order of offers is used in the single-offer definition (Theorem 1) and of whether each step involves a single offer or multiple offers (Theorem 2). Among the substitutability conditions proposed to date, bilateral substitutability (in conjunction with irrelevance of rejected contracts) is the weakest sufficient condition to guarantee the feasibility of the COP's outcome.[2] Thus, our results would cover most of the cases in which the COP is known to be relevant.[3]

The rest of the note is organized as follows. Section 2 describes the model. Section 3 presents the two definitions of the COP. Section 4 establishes the results.

2. Preliminaries

We study the standard setting of a many-to-one matching market with contracts. Let D and H be finite sets of doctors and hospitals, respectively. The finite set of possible contracts is given by $X \subseteq D \times H \times \Theta$ for some Θ.[4] For an arbitrary contract $x \in X$, let $[x]_D$ be its projection onto D, i.e., $[x]_D = d$ if and only if $x = (d, h, \theta)$ for some $h \in H$ and $\theta \in \Theta$. Similarly, for any $X' \subseteq X$, let $[X']_D := \{d \in D : d = [x]_D \text{ for some } x \in X'\}$. For the hospital side, $[x]_H$ and $[X']_H$ are analogously defined. A subset $X' \subseteq X$ of contracts is said to be a (feasible) *allocation* if $x, x' \in X'$ and $x \neq x'$ imply $[x]_D \neq [x']_D$.

Each doctor $d \in D$ has a strict preference relation, denoted by $\succ_d$, over $\{x \in X : [x]_D = d\} \cup \{\varnothing\}$, where $\varnothing$ represents d's outside option and is referred to as a *null contract*.[5] The profile of the doctors' preferences is denoted by $\succ_D = (\succ_d)_{d\in D}$. A non-null contract x is said to be *acceptable* to doctor d if $x \succ_d \varnothing$.[6] The set of acceptable contracts to doctor d is given by

$$\mathrm{Ac}(\succ_d) := \{x \in X : [x]_D = d \text{ and } x \succ_d \varnothing\}.$$

Without loss of generality, assume $\mathrm{Ac}(\succ_d) \neq \emptyset$ for all $d \in D$. Each hospital $h \in H$ has a choice function $C_h : 2^X \to 2^X$ such that (i) $C_h(X') \subseteq X'$, (ii) $[C_h(X')]_H \subseteq \{h\}$, and (iii) $[x, y \in C_h(X') \text{ and } x \neq y] \Rightarrow [x]_D \neq [y]_D$. The profile of the hospitals' choice functions is denoted by $C_H = (C_h)_{h\in H}$. We will impose the following two conditions on the choice functions.

Definition 1 (*Hatfield and Kojima, 2010*)**.** Hospital h's choice function $C_h(\cdot)$ satisfies the *bilateral substitutability* condition if there do not exist contracts $x, y \in X$ and a subset $X' \subseteq X$ of contracts such that $[x]_D, [y]_D \notin [X']_D$, $x \notin C_h(X' \cup \{x\})$, and $x \in C_h(X' \cup \{x, y\})$. □

Definition 2 (*Aygün and Sönmez, 2013*)**.** Hospital h's choice function $C_h(\cdot)$ satisfies the *irrelevance of rejected contracts* (henceforth, IRC) condition if $x \notin C_h(X' \cup \{x\})$ implies $C_h(X' \cup \{x\}) = C_h(X')$ for all $X' \subset X$ and $x \in X \setminus X'$. □

In the presence of IRC, we can rewrite the requirement of bilateral substitutability as follows.[7]

Lemma. *Suppose that hospital h's choice function $C_h(\cdot)$ satisfies the bilateral substitutability condition and the IRC condition. For any $d, d' \in D$, $x \in X$, and $X' \subseteq X$, then, $d, d' \notin [C_h(X')]_D$, $d \neq d'$, and $[x]_D = d$ imply $d' \notin [C_h(X' \cup \{x\})]_D$.*

Proof. Suppose that $d, d' \notin [C_h(X')]_D$, $d \neq d'$, and $[x]_D = d$. Towards a contradiction, suppose also that there exists $x' \in C_h(X' \cup \{x\})$ with $[x']_D = d'$. Note that this implies $x \in C_h(X' \cup \{x\})$, because otherwise $C_h(X' \cup \{x\}) = C_h(X')$ must hold by IRC. Define $X'' := X' \setminus \{y \in X' : [y]_D \in \{d, d'\}\}$. We then make three observations: First, $d, d' \notin [X'']_D$ by construction. Second, as $d, d' \notin [C_h(X')]_D$, IRC implies $C_h(X'' \cup \{x'\}) = C_h(X') \not\ni x'$. Third, since $x, x' \in C_h(X' \cup \{x\})$, IRC also implies $C_h(X'' \cup \{x, x'\}) = C_h(X' \cup \{x\}) \ni x'$. These, however, contradict the assumption of bilateral substitutability. ■

3. Cumulative Offer Process(es)

As mentioned in the introduction, we will consider two definitions of the COP. We first introduce the one à la Hatfield and Kojima (2010).

Definition 3 (*Hatfield and Kojima, 2010*)**.** A *(single-offer) COP* proceeds as follows.

- Step 1: One arbitrarily chosen doctor, d_1, offers her first choice contract x_1. Let $P_1 = \{x_1\}$. The hospital that is offered the contract, $h_1 = [x_1]_H$, holds the contract if $C_{h_1}(\{x_1\}) = \{x_1\}$ and rejects it otherwise. Let $D_1 = D \setminus \{d_1\}$ if h_1 holds x_1, and $D_1 = D$ otherwise.
- Step $t \geq 2$: One doctor, d_t, arbitrarily chosen from D_{t-1} offers her best contact, x_t, among those that have not been offered (i.e., among $X \setminus P_{t-1}$). Let $P_t = P_{t-1} \cup \{x_t\}$ be the pool of offers that have been offered up to this step. Among P_t, each hospital h holds the best combination of contracts, $C_h(P_t)$. Finally, let D_t be the set of doctors for whom (i) no contract is currently held by any hospital and (ii) not all acceptable contracts have been offered yet, i.e.,

$$D_t = \{d \in D : d \notin [C_h(P_t)]_D \text{ for all } h \in H \text{ and } \mathrm{Ac}(\succ_d) \setminus P_t \neq \emptyset\}. \quad (1)$$

  Proceed to step $t + 1$ if D_t is non-empty and terminate otherwise.
- Outcome: When the process terminates at step T, its outcome is $\bigcup_{h\in H} C_h(P_T)$. □

Note that the above definition does *not* pin down a unique algorithm, because it does not fully specify who should make an offer at each step when there exist multiple doctors for whom no contract is held. Hence, according to Definition 3, there exist multiple COPs, even though $(\succ_D, C_H)$ is fixed. Our purpose is to show that all of those COPs induce the same final outcome and thus, they can be seen as a single algorithm (Theorem 1). To do so, we will identify a COP with the sequence of contracts that are offered during the process. This identification enables us to analyze any process accommodated in Definition 3 without directly specifying the order in which the doctors make offers.

Definition 4. A finite sequence of contracts $(x_t)_{t=1}^T$ is said to *represent a COP* at $(\succ_D, C_H)$, if it satisfies all of the following conditions:

- For all $t \in \{1, \ldots, T\}$ and $h \in H$, it holds that $[x_t]_D \notin [C_h(\{x_1, \ldots, x_{t-1}\})]_D$.
- For all $t \in \{1, \ldots, T\}$ and $y \in X$, $[x_t]_D = [y]_D = d$ and $y \succ_d x_t$ imply $y = x_\tau$ for some $\tau < t$.
- For all $d \in D$, either (i) there exists $h \in H$ such that $d \in [C_h(\{x_1, \ldots, x_T\})]_D$, or (ii) $\mathrm{Ac}(\succ_d) \subseteq \{x_1, \ldots, x_T\}$.

The *outcome* induced by $(x_t)_{t=1}^T$ is $\bigcup_{h\in H} C_h(\{x_1, \ldots, x_T\})$. □

An alternative way to define the COP, which is equivalent to Hatfield and Milgrom's (2005) original definition, is to let all the doctors with no contract temporarily held simultaneously make an offer at each step. Note that the following definition unambigu-

[2] See Hatfield and Kojima (2010). See also Aygün and Sönmez (2013), who point out the hidden assumption of irrelevance of rejected contracts.

[3] Kominers and Sönmez (2013, Theorem B.1) show that the COP is order-independent if hospitals' choice functions are generated by slot-specific priorities. For such choice functions always satisfy the two conditions we impose, our Theorems 1 and 2 can be seen as a generalization of their result.

[4] For example, Θ can be interpreted as the set of possible wage levels (Kelso and Crawford, 1982) and/or job descriptions (Roth, 1984).

[5] To avoid confusion, the empty set will be denoted by $\emptyset$.

[6] Note that $x \succ \varnothing$ implies $[x]_D = d$ since $\succ_d$ is defined over $\{x \in X : [x]_D = d\} \cup \{\varnothing\}$.

[7] The following Lemma is essentially a summary of the arguments that Hatfield and Kojima (2010, pp. 1710–11) use to show the outcome of the COP is feasible under the two assumptions.