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Computing the risky steady state of DSGE models

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HIGHLIGHTS

- The risky steady state incorporates important information above future expected risk.
- This note describes a direct method for solving for the risky steady state.
- The procedure is fast and accurate for small perturbations.

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ABSTRACT

This note describes a simple procedure for solving the *risky steady state* in medium-scale macroeconomic models. This is the "point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date" [Coeurdacier, N., Rey, H., Winant, P., 2011. The risky steady state. The American Economic Review 101 (3), 398–401]. This new procedure is a direct method which makes use of a second-order approximation of the macroeconomic model around its *deterministic* steady state, thus avoiding the need to employ an iterative algorithm to solve a fixed-point problem.

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1. Introduction

Coeurdacier et al. (2011) propose the concept of the *risky steady state* as a preferable alternative to the commonly used deterministic steady state, around which to approximate the dynamics of macroeconomic models. This concept is particularly useful in models with portfolio choice problems.[1] This is because, as the name suggests, it incorporates information about the stochastic nature of the economic environment. It is also a useful candidate for comparing the welfare implications of alternative macroeconomic policy tools. The risky steady state, however, is computationally more demanding than its deterministic counterpart, precisely because it requires the steady state and the corresponding dynamics to be jointly determined. The established procedure is to use an iterative method to find the fixed point of a problem in which one function maps steady states into second moments and another maps second moments back into steady states.[2] The difficulty is that this iterative method becomes more complex as one increases the dimensions of the model. This note proposes an alternative method which dispenses with the fixed-point problem. Instead, it makes use of the second-order accurate approximation of the model around its deterministic steady state, as the first step in a two-step process to compute the risky steady state. This new procedure is computationally less demanding and only suffers from a minimal loss in accuracy of the approximation.

2. The model

The competitive equilibrium conditions of a medium-scale macroeconomic model can be written as

$$E_t\left[f\left(y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, z_t\right)\right] = 0 \qquad (1)$$

$$z_{t+1} = \Lambda z_t + \eta\sigma\varepsilon_{t+1},$$

where y_t is an $n_y \times 1$ vector of endogenous nonpredetermined variables, x_t is an $n_x \times 1$ vector of endogenous predetermined

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[1] Coeurdacier et al. (2011) use a small open-economy model for illustration. Gertler et al. (2012) make use of the risky steady state for a financial accelerator model in which banks face a portfolio choice problem on the liability side of their balance sheets.

[2] Juillard (2011) extends the procedure to a second-order approximation of the decision rule.

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variables, z_t is an $n_z \times 1$ vector of exogenous variables, and ε_t is an $n_z \times 1$ vector of exogenous i.i.d. innovations with mean zero and unit standard deviations. The matrices Λ and η are of order $n_z \times n_z$, and $\sigma > 0$ is a scalar scaling the amount of uncertainty in the economy.

Let the decision rules that solve the system of equations in (1) be $y_t = g(x_t, z_t, \sigma)$ and $x_{t+1} = h(x_t, z_t, \sigma)$. Using superscripts d and r to denote variables at the deterministic and risky steady state, respectively, x^d solves $x^d = h\left(x^d, 0, 0\right)$ and we have the following definition.

Definition 1. The risky steady state x^r solves

$$x^r = h\left(x^r, 0, \sigma\right) \quad \text{with } y^r = g\left(x^r, 0, \sigma\right). \tag{2}$$

This definition of the risky steady state formalizes the definition in Coeurdacier et al. (2011) (quoted in the abstract above). Solving for the deterministic steady state is comparatively easy since $\left(y^d, x^d\right)$ also solves $f\left(y^d, y^d, x^d, x^d, 0, 0\right) = 0$. In contrast, the risky steady state (y^r, x^r) solves $Ef(y_{t+1}, y^r, x^r, x^r, z_{t+1}, 0) = 0$, whose computation is complicated by the existence of the expectations operator. Solving the risky steady state therefore requires knowledge of the unknown decision rules g and h.

3. The theory

3.1. Risky steady state

Following Schmitt-Grohé and Uribe (2004), a second-order accurate approximation to the decision rule h in the neighbourhood of the deterministic steady state is

$$x^i_{t+1} = x^{d,i} + h^{d,i}_x\left(x_t - x^d\right) + \frac{1}{2}\left(x_t - x^d\right)' h^{d,i}_{xx}\left(x_t - x^d\right) + \frac{1}{2} z_t' h^{d,i}_{zz} z_t + \left(x_t - x^d\right)' h^{d,i}_{xz} z_t + \frac{1}{2} h^{d,i}_{\sigma\sigma}\sigma^2, \tag{3}$$

where $h^{d,i}_x$, $h^{d,i}_{xx}$, $h^{d,i}_{xz}$, $h^{d,i}_{zz}$, and $h^{d,i}_{\sigma\sigma}$ are matrices of first and second partial derivatives, with the d superscripts indicating that these objects have all been evaluated at the deterministic steady state. To find an approximation for the risky steady state using Eq. (3), the definition of the risky steady state in Eq. (2) is applied. This involves setting $x_{t+1} = x_t = x^r$ and $z_t = 0$ in Eq. (3) to give

$$x^{r,i} = x^{d,i} + h^{d,i}_x\left(x^r - x^d\right) + \frac{1}{2}\left(x^r - x^d\right)' h^{d,i}_{xx}\left(x^r - x^d\right) + \frac{1}{2} h^{d,i}_{\sigma\sigma}\sigma^2. \tag{4}$$

Stacking each of the $i = 1, \ldots, n_x$ decision rules in equation (4), and defining $x^* \equiv x^r - x^d$ as the deviation between the risky and deterministic steady state, leaves a matrix quadratic

$$0 = C + Bx^* + A\,\mathrm{vec}\left(x^* x^{*\prime}\right)$$

to be solved, where

$$C \equiv h^d_{\sigma\sigma}\frac{\sigma^2}{2}, \qquad B \equiv h^d_x - I_{n_x}, \qquad A \equiv \frac{1}{2}\begin{bmatrix} \mathrm{vec}\left(h^{d,1}_{xx}\right)' \\ \vdots \\ \mathrm{vec}\left(h^{d,n_x}_{xx}\right)' \end{bmatrix}.$$

Matlab code to implement this procedure is available from the author's homepage; it makes use of code from Schmitt-Grohé and Uribe (2004).[3]

[3] http://sites.google.com/site/oliverdegroot/research. Solving the matrix quadratic makes use of csolve.m.

3.2. First-order dynamics

With an approximation of the risky steady state in hand, it is possible to solve for a first-order approximation of the decision rules g and h in the neighbourhood of the risky steady state. But, to do this, one also needs knowledge of the first derivatives of $E_t f(\cdot)$ evaluated at the risky steady state. Since it is not possible to evaluate these derivatives exactly, a Taylor approximation is used. For example, a zeroth-order Taylor approximation of a first derivative of $E_t f(\cdot)$ at the risky steady state (which I will call *method 1*) is simply to evaluate the first derivative of $E_t f(\cdot)$ at $(y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, z_t) = (y^r, y^r, x^r, x^r, 0, 0)$, which, with the solution (y^r, x^r) in hand, is straightforward.

The procedure to find a second-order approximation of the derivatives of $E_t f(\cdot)$ evaluated at the risky steady state (*method 2*) is a little more involved.[4] First, compute analytically the first derivatives of $E_t f(\cdot)$ and include them as a vector of auxiliary variables and equations in the code of Schmitt-Grohé and Uribe. Second, solving the stacked system (of equilibrium conditions and auxiliary equations) following the method described in the previous subsection delivers a second-order approximation of the first derivatives of $E_t f(\cdot)$ evaluated at the risky steady state. Third, with the first derivatives appropriately evaluated, a first-order approximation of the model's dynamics in the neighbourhood of the risky steady state is then possible using standard linear solution techniques.[5]

4. Application

To illustrate the method described in Section 3, I use the Burnside (1998) asset pricing model, since it provides a closed-form solution for the decision rule g.[6] The model consists of a representative agent who maximizes the lifetime utility function $E_0 \Sigma_{t=0}^{\infty} \beta^t c_t^{\theta}/\theta$ subject to $p_t e_{t+1} + c_t = p_t e_t + d_t e_t$ and a borrowing limit that prevents agents from engaging in Ponzi games. In the above expression, c_t denotes consumption, p_t the relative price of trees in terms of consumption goods, e_t the number of trees owned by the representative household at the beginning of period t, and d_t the dividends per tree in period t. Dividends are assumed to follow an exogenous stochastic process given by $d_{t+1} = \exp(x_{t+1})\, d_t$, where $\exp(x_{t+1})$ denotes the gross growth rate of dividends, and $x_{t+1} = (1-\rho)\bar{x} + \rho x_t + \sigma\varepsilon_{t+1}$, where $\varepsilon_t \sim \mathrm{NIID}(0, 1)$. The optimality conditions associated with the household are the above budget constraint, borrowing limit, and $p_t c_t^{\theta-1} = \beta E_t\left[c_{t+1}^{\theta-1}(p_{t+1} + d_{t+1})\right]$. In equilibrium, $c_t = d_t$ and $e_t = 1$. Defining the price–dividend ratio as $y_t = p_t/d_t$ yields

$$y_t = \beta E_t\left[\exp(\theta x_{t+1})(1 + y_{t+1})\right]. \tag{5}$$

Burnside (1998) shows that the non-explosive solution to this equation is of the form

$$y_t \equiv g(x_t, \sigma) = \sum_{t=0}^{\infty} \beta^i \exp\left(a_i + b_i(x_t - \bar{x})\right),$$

where

$$a_i = \theta\bar{x}i + \frac{\theta^2\sigma^2}{2(1-\rho)^2}\left[i - \frac{2\rho\left(1-\rho^i\right)}{1-\rho} + \frac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right];$$

$$b_i = \frac{\theta\rho\left(1-\rho^i\right)}{1-\rho}.$$

[4] A first-order approximation delivers the same result as the zeroth-order approximation.

[5] Detailed Matlab code and documentation are available from the author.

[6] Burnside (1998) has also been used by Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004), and Juillard (2011).