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Cycles with undistinguished actions and extended Rock–Paper–Scissors games

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HIGHLIGHTS

- We introduce a new class of symmetric zero-sum games based on cycles.
- We find an infinity of Nash equilibria when the number of actions is even.
- We characterize the unique Nash equilibrium when the number of actions is odd.

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ABSTRACT

This paper examines zero-sum games that are based on a cyclic preference relation defined over undistinguished actions. For each of these games, the set of Nash equilibria is characterized. When the number of actions is odd, a unique Nash equilibrium is always obtained. On the other hand, in the case of an even number of actions, every such game exhibits an infinite number of Nash equilibria. Our results give some insights as to the robustness of Nash equilibria with respect to perturbations of the action set.

1. Introduction

We investigate the class of zero-sum games that are based on a cyclic preference relation with undistinguished alternatives. A remarkable member of this family is the *Rock–Paper–Scissors* game, whose unique Nash equilibrium is obtained when both players put the same weight on each of the three actions. See Van den Nouweland (2007) for an elegant proof of this known result.

A related contribution is that of Bahel (2012), who characterizes the unique Nash equilibrium of zero-sum games that are based on a cycle with distinguished alternatives. As we argue further on, two important features (the indistinction of the actions in our cycles and the fact that the payoff in case of a win is variable) clearly differentiate the present family of games from that in Bahel (2012). Consequently, the two methods of analysis and the characterization results obtained differ. A further related contribution is that of Duersch et al. (2012). We call the class of games under investigation here Extended Rock–Paper–Scissors (ERPS) games. Duersch et al. (2012) define a class of "generalized Rock–Paper–Scissors (gRPS) games" that contains the ERPS games as a proper subclass. They show in particular that a symmetric two-player zero-sum game has a Nash equilibrium in pure strategies if and only if it is not a gRPS game. It follows that an ERPS game does not have a Nash equilibrium in pure strategies. We characterize the set of all Nash equilibria, none of which is in pure strategies, for every ERPS game.

Our analysis makes use of the minimax theorem for zero-sum games. It is shown that, when the number of actions available to the players is odd, there is a unique Nash equilibrium in which the players give a positive weight to each available action. It is known from Kaplansky (1945) that an odd number of actions is necessary for a symmetric zero-sum game to exhibit only Nash equilibria that give a positive weight to each action. Our results show that this condition is also sufficient for the subclass of ERPS games. On the other hand, we find that there is a continuum of Nash equilibria when the number of actions is even. In addition, we provide an explicit characterization of the set of Nash equilibria for all ERPS games.

We argue in the discussion of Section 4 that our results help understand the dynamics of populations of animals that exhibit competitive cycles. Our analysis also helps better understand how changes to the action set may affect the set of Nash equilibria in symmetric zero-sum games.

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2. Extended Rock–Paper–Scissors games

Consider a finite set of actions $A = \{a_1, a_2, \dots, a_K\}$ consisting of $K \geq 3$ alternatives.

Definition 1. A binary relation $\succ$, defined over A, will be called a *cycle (with undistinguished actions)* if

$$a_k \succ a_\ell \quad \text{if and only if} \quad k \equiv \ell + 1 \mod K. \tag{1}$$

In Definition 1, we essentially have a cyclic preference relation over A. Note that a_2 is preferred to a_1, a_3 is preferred to $a_2, \dots, a_K$ is preferred to a_{K-1}, but a_1 is preferred to a_K. In addition, we have the following: (a) each action $a_k \in A$ is $\succ$-preferred to exactly one other action in A, (b) there is exactly one other action in A that is $\succ$-preferred to any given $a_k \in A$.

Throughout the paper, for any integer $z \in \mathbb{Z}$, we use the notation $\bar{z}$ to refer to the unique number $x \in \{1, \dots, K\}$ such that $x \equiv z \mod K$.[1] Observe that relabeling each action $a_k \in A$ as $a_{\overline{k+l}}$ (for some fixed $l \in \{1, \dots, K-1\}$) and considering anew the binary relation obtained from Definition 1 would result in exactly the same preference relation.[2] In other words, any of the actions $a_1, a_2, \dots, a_K$ can be seen as the beginning (and the end) of the cycle without it affecting the binary relation: in this sense, we say that the actions of the cycle $\succ$ are (structurally) undistinguished.

The following definition introduces the class of games studied in this paper.

Definition 2. A game $[S_i, u_i]_{i=1,2}$ will be called *an Extended Rock–Paper–Scissors* (ERPS) game if there exists a cycle $\succ$ (defined over $A = \{a_1, \dots, a_K\}$) and a positive vector $\alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}^K_{++}$ such that, for $i \in \{1, 2\}$, $S_i = A$ and player i's payoff is

$$u_i(a_k, a_\ell) = \begin{cases} \alpha_k & \text{if } a_k \succ a_\ell, \\ -\alpha_\ell & \text{if } a_\ell \succ a_k, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

when i chooses $a_k \in A$ and the other player chooses $a_\ell \in A$.

We use the notation $G(\succ, \alpha)$ to refer to an arbitrary ERPS game. In essence, $G(\succ, \alpha)$ is a symmetric zero-sum game such that a player wins (i.e., gets a positive payoff) if and only if he/she picks an action that is $\succ$-preferred to that of his/her opponent. If neither player chooses an action that is $\succ$-preferred to the opponent's action, the outcome is a tie (which results in a payoff of zero for both players). Observe that a player's payoff when he/she wins, α_k, may vary with his/her winning action. Likewise, a player's payoff when he/she loses may vary depending on his/her losing action. This means that, although the cycle $\succ$ treats all actions $a_1, \dots, a_K$ in the same way, these actions are in general not undistinguished in the game $G(\succ, \alpha)$. For the following subclass of ERPS games, however, all actions available to the players receive identical treatment and are therefore undistinguished.

Definition 3. An ERPS game $G(\succ, \alpha)$ will be called a *cycle-based game with undistinguished actions* (CBGU) if $\alpha_k = \alpha_\ell$, for any $a_k, a_\ell \in \{a_1, \dots, a_K\}$.

The families CBGU and ERPS are new to the literature on zero-sum games. In Bahel (2012), $\succ$ is a complete binary relation on A such that $a_k \succ a_\ell$ if $k > \ell$, unless $k = K$ and $\ell = 1$ (in which case $a_1 \succ a_K$).[3] Obviously, action a_1 plays a distinguished role in his setting. Another important difference (with ERPS games) is the fact that, in his framework, a player's payoff to a win (loss) is the same, regardless of the winning (losing) action. In fact, one can easily check that the family ERPS and the class of cycle-based games defined in Bahel (2012) have an empty intersection whenever $K \geq 4$. When $K = 3$, the two families share exactly one element, which (up to the names of the three actions) is the Rock–Paper–Scissors (RPS) game depicted by Example 1.

Next, we provide two examples that illustrate the families of zero-sum games introduced in Definitions 2 and 3, respectively.

Example 1 (*Rock–Paper–Scissors*). [4] Each of two players simultaneously announces either *Rock*, *Paper*, or *Scissors*. *Paper* beats (wraps) *Rock*, *Rock* beats (blunts) *Scissors*, and *Scissors* beats (cuts) *Paper*. The player who names the winning object receives $1 from his/her opponent; if both players make the same choice, then no payment is made.

Observe that the RPS game is based on a cycle of length $K = 3$; its matrix representation is as follows.

		Player 2		
		Rock	*Paper*	*Scissors*
Player 1	*Rock*	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

It is well known that either player plays each of the three actions *Rock*, *Paper*, *Scissors* with the same probability in the unique Nash equilibrium of this game.

Example 2 (*Modified Rock–Paper–Scissors*). Consider the modified version of the RPS game depicted by the following matrix.

		Player 2		
		Rock	*Paper*	*Scissors*
Player 1	*Rock*	0, 0	−1, 1	3, −3
	Paper	1, −1	0, 0	−1, 1
	Scissors	−3, 3	1, −1	0, 0

It can be seen from the matrix that we still have *Paper* beats *Rock*, *Rock* beats *Scissors*, and *Scissors* beats *Paper*. However, the payoff to a win depends on the winning action: unlike in the standard version, a player gets a payoff of $3 if he/she wins with *Rock*.

The above two examples are both ERPS games, but only the first one is a cycle-based game with undistinguished actions. An interesting exercise is to try and predict how changes to the RPS payoff matrix would affect the set of Nash equilibria of the game.[5] A more important issue is to determine whether there is a generic way of deriving the set of Nash equilibria of an arbitrary ERPS game. The next section investigates these questions.

3. Analysis

Let $M_i = \{(m_i(a_1), \dots, m_i(a_K)) \in \mathbb{R}^K_+ \mid \sum_{k=1}^K m_i(a_k) = 1\}$ denote the set of mixed strategies of player $i \in \{1, 2\}$. The number $m_i(a_k)$ stands for the probability that player i plays action a_k, $k = 1, \dots, K$. Player i's expected payoff to any mixed-strategy pair

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1 For instance, $\overline{K+1} = 1$ and $\overline{-1} = K - 1$.

2 Observe that the shift transformation $t_l : k \to \overline{k+l}$ is a permutation of $\{1, \dots, K\}$.

3 Notice from definition (1) that, in our setting, some actions in A are not $\succ$-comparable when $K > 3$.

4 We adopt the language of Osborne (2004, p. 141) for this example. The game has also been described by Von Neumann (1928, p. 303).

5 Weibull (1995, p. 77) considers "generalized Rock–Paper–Scissors games" in which the RPS payoff matrix is modified in such a way that the game is no longer a constant-sum game but is still symmetric, and the Nash equilibrium is invariant under these modifications.