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Equivalence and axiomatization of solutions for cooperative games with circular communication structure

Özer Selçuk, Takamasa Suzuki, Dolf Talman *

CentER, Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

HIGHLIGHTS

- In this paper transferable utility games with limited coalition formation are considered.
- Players are located on a circle and only connected coalitions are feasible.
- As solutions the averages of two classes of marginal vectors are considered.
- On the considered class of games the two solutions coincide.
- The solutions are characterized by independent axioms.

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ABSTRACT

We study cooperative games with transferable utility and limited cooperation possibilities. The focus is on communication structures where the set of players forms a circle, so that the possibilities of cooperation are represented by the connected sets of nodes of an undirected circular graph. Single-valued solutions are considered which are the average of specific marginal vectors. A marginal vector is deduced from a permutation on the player set and assigns as payoff to a player his marginal contribution when he joins his predecessors in the permutation. We compare the collection of all marginal vectors that are deduced from the permutations in which every player is connected to his immediate predecessor with the one deduced from the permutations in which every player is connected to at least one of his predecessors. The average of the first collection yields the average tree solution and the average of the second one is the Shapley value for augmenting systems. Although the two collections of marginal vectors are different and the second collection contains the first one, it turns out that both solutions coincide on the class of circular graph games. Further, an axiomatization of the solution is given using efficiency, linearity, some restricted dummy property, and some kind of symmetry.

1. Introduction

A cooperative game with transferable utility, or TU-game, consists of a finite set of players and a characteristic function that assigns a worth to any subset of players. Players within such a coalition can freely distribute the worth of the coalition as payoff among themselves. The problem of a TU-game is how much payoff each player must receive. One of the most well-known single-valued solutions is the Shapley value (Shapley, 1953) being the average of all marginal vectors of the game. A marginal vector is a payoff vector in which for some permutation on the player set each player receives as payoff his marginal contribution when he joins his predecessors in the permutation. The Shapley value is uniquely characterized by efficiency, the dummy property, linearity, and symmetry.

In many economic situations there exist restrictions which prevent some coalitions from cooperating. Myerson (1977) introduces games with communication structure. Graph games arise when the restriction is represented by a graph in which the nodes of the graph represent the players and a link between two players indicates that they can communicate with each other. Only connected subsets of agents, called networks, are assumed to be able to form a coalition and attain their worth.

The average tree solution, introduced in Herings et al. (2008) on the class of cycle-free graph games, is the average of the marginal

* Corresponding author. Tel.: +31 13 4662346; fax: +31 13 4663280.
E-mail addresses: o.selcuk@tilburguniversity.edu (Ö. Selçuk), t.suzuki@tilburguniversity.edu (T. Suzuki), talman@uvt.nl, talman@tilburguniversity.edu (D. Talman).

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vectors deduced from all spanning trees on the graph. A characterization using unanimity games on this class of graph games is given in Mishra and Talman (2010). In Herings et al. (2010) the average tree solution is generalized to the class of arbitrary graph games by taking the average of the marginal vectors deduced from a specific subclass of spanning trees. On the class of non-cycle-free graph games characterizations using standard axioms or unanimity games are not known in the literature.

In this paper we consider the class of circle graph games where the underlying graph is assumed to be circular. Players could be firms or cities situated along a lake, around a mountain, or on a circular pipeline, where players can only be connected to their direct neighbors, one located on each side. In this setting a set of players is assumed only to be able to cooperate if they form a segment of the circle. We consider two different natural collections of permutations and take as solution concept the average of the deduced marginal vectors. The set of marginal vectors at which players are only able to contribute if they are connected to their immediate predecessor in the corresponding permutation appears to be the set of marginal vectors whose average is the average tree solution on the class of circle graph games, while the other collection at which players are only able to contribute if they are connected to one of their predecessors underlies the Shapley value introduced in Bilbao and Ordóñez (2009) on the class of games on augmenting systems, which includes the class of circle graph games. Although the second set of marginal vectors contains the first set as a proper subset we show that the two solutions give the same payoff vector on the class of circle graph games. We further give for this class an axiomatization of the solution using standard axioms. The solution does not satisfy symmetry, but we show that it is fully characterized by efficiency, linearity, a restricted form of the dummy property, and some weak form of symmetry.

This paper is organized as follows. In Section 2 the class of circle graph games is introduced. In Section 3 several solutions are discussed and it is shown that they coincide on this class of games. In Section 4 an axiomatic characterization is given.

2. Circle graph games

Consider a finite set $N = \{1, \ldots, n\}$ of $n, n \geq 3$, players or agents located on a circle. Without loss of generality let $L = \{\{i, i+1\} \mid i = 1, \ldots, n\}$ denote the set of links between the players, where $i + 1 = 1$ when $i = n$. The pair (N, L) is an undirected circle graph with the player set N as the set of nodes and the link set L as the set of edges. A coalition $S \in 2^N$ is connected, or a network, if for any $i, j \in S$ there is a sequence of different nodes $(i_1, i_2, \ldots, i_k)$ in S such that $i_1 = i$, $i_k = j$ and $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. The collection of networks in (N, L) is denoted $C^L(N)$.

A permutation on the player set N is a bijection on N and represents an order in which the players can join each other to form the grand coalition N of all players. $\Pi(N)$ denotes the set of all permutations on N. A permutation in $\Pi(N)$ is *admissible* if every player is connected to its immediate predecessor in the permutation. Let $\Pi^a(N)$ denote the set of admissible permutations on N, then

$$\Pi^a(N) = \{\sigma \in \Pi(N) \mid \{\sigma(i), \sigma(i+1)\} \in L,\ i = 1, \ldots, n-1\}.$$

For every $i \in N$ there are two admissible permutations σ with $\sigma(1) = i$, denoted by $\sigma_1^i = (i, i+1, \ldots, n, 1, \ldots, i-1)$ and $\sigma_2^i = (i, i-1, \ldots, 1, n, \ldots, i+1)$, and so $|\Pi^a(N)| = 2n$ and $\Pi^a(N) = \{\sigma_1^i \mid i \in N\} \cup \{\sigma_2^i \mid i \in N\}$.

A permutation in $\Pi(N)$ is *compatible* if every player is connected to at least one of its predecessors, not necessarily the immediate predecessor in the permutation. Let $\Pi^c(N)$ denote the set of compatible permutations on N, then

$$\Pi^c(N) = \{\sigma \in \Pi(N) \mid \{\sigma(1), \ldots, \sigma(k)\} \in C^L(N),\\ k = 1, \ldots, n-1\}.$$

For every $i \in N$ there are two choices of $\sigma(2)$ for being compatible with $\sigma(1) = i$, namely $i-1$ and $i+1$, where $i-1 = n$ if $i = 1$ and $i+1 = 1$ if $i = n$. In general, for $k = 2, \ldots, n-1$, there are two choices of $\sigma(k)$ for being compatible with $(\sigma(1), \sigma(2), \ldots, \sigma(k-1))$. Since $\sigma(n)$ is uniquely determined, this leads to $|\Pi^c(N)| = 2^{n-2}n$. Notice that $\Pi^a(N) \subset \Pi^c(N)$.

A characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, defines the worth of a coalition $S \in 2^N$. Following Myerson (1977), only networks in (N, L) are able to distribute freely their worth as payoff among their members. The triple (N, v, L) is called a *circle graph game* and the class of circle graph games is denoted by $\mathcal{G}^c$. The problem of a circle graph game is how much payoff each player must receive. Given a circle graph game $(N, v, L) \in \mathcal{G}^c$, to any permutation $\sigma \in \Pi(N)$ a marginal (contribution) vector $m^\sigma(N, v, L)$ corresponds, assigning to agent $\sigma(k)$, $k = 1, \ldots, n$, as payoff

$$m^\sigma_{\sigma(k)}(N, v, L) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}).$$

3. Equivalent solutions

The average tree solution, introduced in Herings et al. (2008) on the class of cycle-free graph games and generalized in Herings et al. (2010) to the class of arbitrary graph games, considers for circle graph games all n-tuples $B = (B_1, \ldots, B_n)$ of networks in (N, L) satisfying for every $i \in N$ that $i \in B_i$ and there is a unique $j \in N$ such that $\{i, j\} \in L$ and $B_j = B_i \setminus \{i\}$. Let $\mathcal{B}^L$ denote the collection of such admissible n-tuples of networks in (N, L). Given a circle graph game $(N, v, L) \in \mathcal{G}^c$, to any $B \in \mathcal{B}^L$ a marginal vector $m^B(N, v, L)$ corresponds with payoff

$$m_i^B(N, v, L) = v(B_i) - v(B_i \setminus \{i\})$$

for agent $i \in N$. The average tree solution, AT, assigns on the class of circle graph games to any game $(N, v, L) \in \mathcal{G}^c$ the average of the marginal vectors corresponding to all admissible n-tuples of networks in (N, L), i.e.,

$$AT(N, v, L) = \frac{1}{|\mathcal{B}^L|} \sum_{B \in \mathcal{B}^L} m^B(N, v, L).$$

First we show that on the class of circle graph games the average tree solution is the average of the marginal vectors induced by all admissible permutations.

Theorem 3.1. *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that*

$$AT(N, v, L) = \frac{1}{|\Pi^a(N)|} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N, v, L).$$

Proof. Take any $\sigma \in \Pi^a(N)$ and suppose $\sigma = \sigma_1^i$ for some $i \in N$. Let $B_k = \{i, \ldots, k\}$ for $k = i, \ldots, n$ and $B_k = \{i, \ldots, n, 1, \ldots, k\}$ for $k = 1, \ldots, i-1$. Then $B = (B_1, \ldots, B_n) \in \mathcal{B}^L$ and $m^B(N, v, L) = m^{\sigma_1^i}(N, v, L)$. Similarly, when $\sigma = \sigma_2^i$ for some $i \in N$, let $B_k = \{k, \ldots, i\}$ for $k = 1, \ldots, i$ and $B_k = \{k, \ldots, n, 1, \ldots, i\}$ for $k = i+1, \ldots, n$. Then again $B = (B_1, \ldots, B_n) \in \mathcal{B}^L$ and $m^B(N, v, L) = m^{\sigma_2^i}(N, v, L)$.

Conversely, take any $B = (B_1, \ldots, B_n) \in \mathcal{B}^L$. There exists unique $i \in N$ such that $B_i = N$. Then either $B_i \setminus \{i\} = B_{i+1}$ (B_1 if $i = n$) or B_{i-1} (B_n if $i = 1$). Suppose $B_i \setminus \{i\} = B_{i+1}$. Then, when $i < n$, $i+2$ (1 if $i = n-1$) is the only element of $B_{i+1} \setminus \{i+1\}$ that is linked in L to $i+1$ and so $B_{i+1} \setminus \{i+1\} = B_{i+2}$, and, when $i = n$, 2 is the only element of $B_1 \setminus \{1\}$ that is linked to 1. In general, for $k = i+1, \ldots, n, 1, \ldots, i$ only element $k+1$ in $B_k \setminus \{k\}$ is linked to k and so $B_k \setminus \{k\} = B_{k+1}$. From this it follows that $m^B(N, v, L) = m^{\sigma_2^{i-1}}(N, v, L)$. Similarly, if $B_i \setminus \{i\} = B_{i-1}$, it holds that $m^B(N, v, L) = m^{\sigma_1^{i+1}}(N, v, L)$. Thus every admissible n-tuple of networks in (N, L) corresponds to a unique admissible permutation in $\Pi^a(N)$. $\square$