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Greediness and equilibrium in congestion games

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HIGHLIGHTS

- Examples that in the general CG the connection between greedy and NE strategy profile sets is arbitrary.
- Proof that in Extension parallel Congestion games there is an equivalence between the greedy and NE strategy profiles.
- Proof that this is the broadest class where this equivalence holds.

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ABSTRACT

We study the class of congestion games for which the set of Nash equilibrium is equivalent to the set of strategy profiles played by greedy myopic players. We show these two coincide iff such games are played over extension-parallel graphs.

1. Introduction

Congestion games form a natural class of games that are useful in modeling many realistic settings, such as traffic and communication networks, routing, load balancing and more. A symmetric congestion game is a 4-tuple $(N, R, \Sigma, \{\mu_r\}_{r\in R})$, where N is a finite set of players, R is a finite set of resources, $\Sigma \subset 2^R$ is the set of players' strategies, and for any $r \in R$, $\mu_r : N \to \mathbb{R}$ is the resource's payoff function. A strategy of player i is a choice of a subset of resources, $s^i \in \Sigma$. For any strategy tuple $s = (s^i)_{i\in N} \in \Sigma^N$ let $c(s)_r = |\{i \in N : r \in s^i\}|$ denote the number of players that utilize r (a.k.a. the congestion of the resource r) and denote by $c(s) = (c(s)_1, \ldots, c(s)_r)$ the congestion vector. The utility of a player is the total payoff for the resources she utilizes. Formally, $U^i(s) = \Sigma_{r\in s^i}\mu_r(c(s)_r)$.

A congestion game is *monotone* if for any $1 \le k < l \le N$ and $r \in R$, $\mu_r(k) \ge \mu_r(l)$. Monotone congestion games widely prevail in modeling traffic and communication problems, production resource allocation and more. In *single-signed* congestion games payoffs are either all positive or all negative. Typically, whenever monotone congestion games are used for modeling, they are assumed single-signed.

A *congestion game form* is a pair $F = \{R, \Sigma\}$, composed of the set of resources and a set of strategies (subsets of R). For any congestion game $G = (N, R, \Sigma, \{\mu_r\}_{r\in R})$ let $F(G) = (R, \Sigma)$ denote the corresponding game form. Given a congestion game form F, let $\mathcal{G}(F) = \{G : (F(G) = F) \wedge (G \text{ monotone})\}$ denote the class of all monotone congestion games with the game form F.

We say that a strategy set, $\Sigma \subset 2^R$, is *subset-free* if for any $s \neq t \in \Sigma$ we have $s \not\subset t$. Thus, a *subset-free Congestion Game (Form)* is a Congestion Game (Form) with a subset-free strategy space. For any equilibrium analysis of single-signed monotone congestion games the assumption of subset-free strategy sets is without loss of generality. In particular, note that in such games for any pair of strategies $s \subset t$ in Σ either s is dominated by t (in case resource payoffs are all positive) or t is dominated by s (in case resource payoffs are all negative) and so after deletion of dominated strategies we are left with subset-free sets.

As usual, a profile $s \in \Sigma^N$ is a pure NE of G, if for each player i, for each strategy $t^i \in \Sigma$, $U^i(s^i, s^{-i}) \ge U^i(t^i, s^{-i})$, where s^{-i} is the vector of strategies of all players but i. Informally, a set of strategies is a Nash equilibrium (NE) if no player can do better by unilaterally

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deviating. The set of all pure NE of a congestion game G will be denoted by $NE(G)$.

Congestion games were introduced by Rosental (1973), who proved that any congestion game has a Nash equilibrium in pure strategies. Albeit natural, the notion of a Nash equilibrium has received much criticism as a realistic outcome of a game. More particularly, for an arbitrary game, an epistemic analysis shows that the conditions needed for such an outcome to prevail are quite strong. In fact, no less than common knowledge of the full game structure on the one hand and common knowledge of rationality on the other hand are required.[1] As for congestion games, which are well known to obtain an, for example, see Monderer and Shapley (1996), one might suspect less is needed. However, this is not known and it may be the case that a Nash equilibrium can be supported by a hierarchy of beliefs regarding players' rationality of finite order.

Fotakis et al. (2006) introduce the notion of a *greedy strategy profile*. Let us consider a dynamic setting with the players joining the game sequentially. Each player, upon arrival, irrevocably chooses a best reply strategy, given the choice of strategies of the previous players, while ignoring subsequent players. The resulting strategy profile is called a *greedy strategy profile*.[2]

Let us denote by $Z(G)$ the set of all greedy strategy profiles. This set is not necessarily a singleton due to the two degrees of freedom in the process—the order of the players and the tie breaking rule in case of indifference among several options. Formally, $s \in Z(G)$, if there exists a permutation $\pi : N \to N$ (one-to-one and onto) of the players ($\pi(i)$ denotes the order of i) such that for any player i who chooses strategy s^i we have $\sum_{r \in s^i} \mu_r(c(s^i_\pi)_r + 1) \geq \sum_{r \in t} \mu_r(c(s^i_\pi)_r + 1)\ \forall t \in \Sigma$, where $c(s^i_\pi)_r = |\{j : r \in s^{\pi(j)}, \pi(j) < \pi(i)\}|$ is the number players preceding i, according to the permutation π, whose strategy includes resource r. Clearly $Z(G) \neq \emptyset$, and typically $Z(G)$ may contain many such profiles, as generally a player may be indifferent between some choices. Let τ be a tie breaking rule, which prescribes a unique choice whenever a player is indifferent between several options. Together, π and τ impose a unique greedy strategy profile on a game G.

In contrast with the rationality assumption underlying the notion of Nash equilibrium, the rationality requirement from a greedy profile is minimal. Indeed players are only assumed to be rational, know their own payoffs and observe the choice of actions by some subset of players (their predecessors). Beyond that nothing is assumed. In particular players need not know whether others are rational or in fact if there any other players in the game beyond the subset of players they observe.

Fotakis et al. (2006) have already shown that $Z(G) \subset NE(G)$ for simple congestion games, where $\Sigma = R$ and Fotakis (2010) extends this congestion games over 'extension-parallel graphs' (in the sequel we formally define this notion) for which the resource payoff functions satisfy a certain property he calls the 'Common Best Reply' (which is trivially satisfied in symmetric congestion games which is the focus of our work). Ackermann et al. (2006) observe that some greedy best response sequences converge very fast to a NE, when the strategy structure is that of a Matroid.

1.1. Our contribution

This paper characterizes the game forms for which $Z(G)$ and $NE(G)$ coincide. In particular, our main result argues that a necessary and sufficient condition for these two solution concepts to coincide is that the game form is that identified in the work of Fotakis (2010), namely 'extension-parallel graphs'. Thus, the marginal contribution of this work over the existing literature is two-fold. First, it is shown that for the game forms in discussion not only is every greedy profile a Nash equilibrium but also vice versa. In addition, we show that for such equivalence to hold for a given game form it must be the case that the game form is of a certain class, namely a 'extension-parallel graph'. In particular, given a game form not satisfying this condition, we show how to construct resource payoff functions such that the set of NE profiles and greedy profiles will not coincide.

Our technical observation regarding necessary and sufficient conditions for which $Z(G)$ and $NE(G)$ shed some light on two aspects of Nash equilibrium in such congestion games:

- The prevalence of a Nash equilibrium outcome—As discussed above the epistemic conditions for the prevalence of an outcome in $Z(G)$ are quite weak compared with those required for the prevalence of an outcome in $NE(G)$ for an arbitrary game, or in fact an arbitrary congestion game. Given our results, in the class of games we study a Nash equilibrium is supported by weak epistemic requirements.[3]
- The speed of convergence to a Nash equilibrium outcome—The number of steps to compute a Nash equilibrium outcome in the class of games we discuss is N (the number of players). This is strictly faster than polynomial or even exponential results (in the number of players) that hold for larger classes of games. We refer the reader to Fotakis (2010) who discusses this in more depth.[4] For more detailed results on the speed of convergence in broader classes of games we refer the reader to Ieong et al. (2005), Ackermann et al. (2006) and Fabrikant et al. (2004). Our results are discouraging for those who assume that such speedy convergence occurs for a broader class of games than those played over extension-parallel game forms.

Interestingly, Holzman and Law Yone (2003), prove that the 'extension-parallel graph' game form is also the necessary and sufficient condition for the set of NE profiles to coincide with the set of strong equilibrium profiles.[5] Combining these results with our contribution we obtain equivalence between greedy profiles and strong NE for the class of 'extension-parallel graph' game forms. Moreover, for game forms outside this class there are resource payoffs function for which this equivalence no longer holds.

The structure of the article is as follows: Section 2 is a 'warm-up' section with a variety of examples that demonstrate that without any restrictions on the game form there is no structural connection between the sets $NE(G)$ and $Z(G)$. Section 3 formalizes the notion of extension parallel games, and discusses the characteristics of this class. Then in Section 4 we present and prove the main result, namely equivalence between $NE(G)$ and $Z(G)$ for extension parallel congestion games.

2. Examples

Here we provide several examples for the various relations between $Z(G)$ and $NE(G)$. As we shall demonstrate those can differ depending on the game in question.

[1] For example, see Aumann and Brandenburger (1995).

[2] There are two similar yet different notions of greediness in the literature on congestion games. The first notion models a situation where players are present in the game and sequentially best-reply to the current game (e.g., Fabrikant et al., 2004 and Fotakis, 2010). The second, which is the one we adopt here, players are initially absent, yet arrive sequentially and, as before, take a best reply to the game being played by the subset of players preceding them. Fotakis et al. (2006) refer to the latter dynamics as a "greedy best response".

[3] One should make a distinction between the rationality assumptions needed to make a NE stable which are typically weak (rationality and knowledge of own payoffs suffices for that) with the rationality assumptions need for players to reach a NE outcome via an introspective analysis, which is what we refer to as the *prevalence* of a Nash equilibrium outcome.

[4] Fotakis (2010) focuses on players that best reply within a game, as opposed to our model where players join the game and best-respond. For the related discussion of the speed of convergence this hardly matters.

[5] Formally, Holzman and Law Yone (2003) introduce a notion of 'tree representable' game forms which are equivalent to extension parallel game forms.