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A switching model with flexible threshold variable: With an application to nonlinear dynamics in stock returns

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HIGHLIGHTS

- Extension of threshold regression model to allow for a flexible threshold variable.
- Threshold variable parameterized as a linear combination of exogenous variables.
- Least squares estimator for the parameters of the model.
- Validity of the proposed methodological framework assessed by a Monte Carlo study.
- Application to nonlinear dynamics in US stock returns.

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ABSTRACT

This paper proposes an extension to threshold-type switching models that lets the threshold variable be a linear combination of exogenous variables with unknown coefficients. An algorithm to estimate the model's parameters by least squares is provided and the validity of the methodological framework is assessed by a Monte Carlo study. The empirical usefulness of the proposed specification is illustrated by an application to US stock returns.

1. Introduction

The data generating processes of several economic and financial variables are characterized by a finite number of states or regimes, which can be described by complementary econometric models. Stock returns are an example in this respect: Paye and Timmermann (2006) show that returns' processes are subject to structural breaks; Guidolin and Timmermann (2006) document the existence of recurrent states linked to the business cycle; and Guidolin et al. (2009) employ Hansen's (2000) threshold regression to capture nonlinear feedback effects. This paper focuses on threshold models, in which the prevailing regime depends on a single threshold variable: this variable has then to be *a priori* chosen or selected according to statistical criteria. This parameterization may however be too restrictive for empirical purposes, as in practice several variables may drive regimes dynamics. In this paper we propose a simple extension to threshold models by letting the threshold variable be a linear combination of a set of variables with unknown coefficients. In our view, this extension can be potentially useful in applied work, such as the analysis of stock returns dynamics.

The paper is organized as follows: Section 2 describes the model; a sufficient identification condition and an estimation algorithm are introduced in Section 3; a Monte Carlo analysis is performed in Section 4; consistently with the previous discussion, an application to US stock returns is provided in Section 5; and concluding remarks are given in Section 6.

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2. Model

We consider the model

$$y_t = \boldsymbol{\beta}'\mathbf{x}_{t-s_x} + \boldsymbol{\delta}'\mathbf{x}_{t-s_x}\mathbf{I}\left(\boldsymbol{\lambda}'\mathbf{q}_{t-s_q} > \alpha\right) + u_t,$$
$$t = s+1, \ldots, T, s \equiv \max\left(s_x, s_q\right), \qquad (1)$$

where $\mathbf{I}(\cdot)$ is the indicator function; $y_t \in \mathcal{Y} \subseteq \mathfrak{R}$ is the dependent variable; $\mathbf{x}_{t-s_x} \in \mathcal{X} \subseteq \mathfrak{R}^{k_x}$ is a $k_x \times 1$ vector of explanatory variables; the threshold variable $\boldsymbol{\lambda}'\mathbf{q}_{t-s_q}$ is a linear combination of the elements of the $k_q \times 1$ vector of random variables $\mathbf{q}_{t-s_q} \equiv \left(q_{1,t-s_q}, \ldots, q_{k_q,t-s_q}\right)' \in \mathcal{Q} \subseteq \mathfrak{R}^{k_q}$ with coefficients collected in the $k_q \times 1$ vector of parameters $\boldsymbol{\lambda} \equiv \left(\lambda_1, \ldots, \lambda_{k_q}\right)'$; α is the threshold value; $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ are $k_x \times 1$ vectors of slope coefficients; u_t is the error term such that $E\left(u_t|\mathbf{x}_{t-s_x}, \mathbf{q}_{t-s_q}\right) = 0$. For the purpose of this paper, the delay parameters $s_x \geq 0$ and $s_q \geq 0$ are assumed to be known and common across the elements of $\mathbf{x}_{t-s_x}$ and $\mathbf{q}_{t-s_q}$, respectively.

The specification in (1) extends Hansen's (2000) generalization of Tong's (1983) threshold autoregressive model by letting the threshold variable $\boldsymbol{\lambda}'\mathbf{q}_{t-s_q}$ be a linear combination of the variables included in $\mathbf{q}_{t-s_q}$ with unknown coefficients collected in $\boldsymbol{\lambda}$. The flexible parameterization $\boldsymbol{\lambda}'\mathbf{q}_{t-s_q}$ in (1) requires neither *a priori* choice of a unique threshold variable from the elements of $\mathbf{q}_{t-s_q}$ nor a selection according to statistical criteria: the threshold variable in (1) reduces to $q_{j,t-s_q}$ when $\boldsymbol{\lambda}$ is the $k_q \times 1$ vector with the j-th element equal to one and all other elements equal to zero. The model in (1) differs from Seo and Linton's (2007) as in the latter the threshold value (and not the threshold variable) is a linear combination of exogenous variables and the threshold variable is *a priori* chosen or selected from the elements of $\mathbf{q}_{t-s_q}$: Seo and Linton's (2007) model is identified without further restrictions, unlike the model in (1) as discussed in Section 3.1.

3. Identification and estimation

3.1. Identification

Without suitable restrictions imposed on $\boldsymbol{\lambda}$ in (1) the parameter vectors $\left(\boldsymbol{\lambda}', \alpha\right)'$ and $\left(h\boldsymbol{\lambda}', h\alpha\right)'$ are observationally equivalent for $0 < h < \infty$: formally, given any $h \in \mathfrak{R}$,

$$P\left[\mathbf{I}\left(\boldsymbol{\lambda}'\mathbf{q}_{t-s_q} > \alpha\right) = \mathbf{I}\left(h\boldsymbol{\lambda}'\mathbf{q}_{t-s_q} > h\alpha\right)\right] = 1 \Leftrightarrow 0 < h < \infty,$$

and (1) is not identified. In Theorem 3.1 below we provide a *sufficient* identification condition that is functional to the empirical application in Section 5: we want to stress this point as less stringent conditions may be derived.[1] Let $\boldsymbol{\iota}_{k_q}$ denote the $k_q \times 1$ vector of ones, and consider the following theorem:

Theorem 3.1. *Let $\boldsymbol{\lambda}'\boldsymbol{\iota}_{k_q} = c$ where c is a known and positive constant. Then the model in (1) is identified.*

Proof of Theorem 3.1. Since $\boldsymbol{\lambda}'\boldsymbol{\iota}_{k_q} = c$ we can write $\lambda_1 = c - \sum_{j=2}^{k_q} \lambda_j$. It follows that

$$\begin{aligned}
\boldsymbol{\lambda}'\mathbf{q}_{t-s_q} &= q_{1,t-s_q}\lambda_1 + \sum_{j=2}^{k_q} q_{j,t-s_q}\lambda_j \\
&= q_{1,t-s_q}c + \sum_{j=2}^{k_q}\left(q_{j,t-s_q} - q_{1,t-s_q}\right)\lambda_j > \alpha \\
&\Leftrightarrow q_{1,t-s_q} > \frac{\alpha}{c} - \sum_{j=2}^{k_q}\left(q_{j,t-s_q} - q_{1,t-s_q}\right)\frac{\lambda_j}{c}.
\end{aligned}$$

Given any $h \in \mathfrak{R}$,

$$\begin{aligned}
&P\left\{\mathbf{I}\left[q_{1,t-s_q} > \frac{\alpha}{c} - \sum_{j=2}^{k_q}\left(q_{j,t-s_q} - q_{1,t-s_q}\right)\frac{\lambda_j}{c}\right]\right. \\
&\quad \left. = \mathbf{I}\left[q_{1,t-s_q} > \frac{h\alpha}{c} - \sum_{j=2}^{k_q}\left(q_{j,t-s_q} - q_{1,t-s_q}\right)\frac{h\lambda_j}{c}\right]\right\} = 1 \\
&\quad \Leftrightarrow h = 1,
\end{aligned}$$

and the model in (1) is identified. □

3.2. Estimation

Given $E(u_t|\mathbf{x}_{t-s_x}, \mathbf{q}_{t-s_q}) = 0$, the parameter vectors $(\boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\lambda}', \alpha)'$ in (1) can be consistently estimated by least squares provided the memory of the sequence $\left\{\mathbf{x}_{t-s_x}, \mathbf{q}_{t-s_q}, u_t\right\}_{t=s+1}^{T}$ is suitably bounded, the required higher order moments exist, and multicollinearity issues are ruled out: see Hansen (2000) for further details. Formally, the least squares estimator for $\left(\boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\lambda}', \alpha\right)'$ is

$$\left(\hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\delta}}', \hat{\boldsymbol{\lambda}}', \hat{\alpha}\right)' = \arg\min_{(\mathbf{b}, \mathbf{d}, \mathbf{l}, a)} \sum_{t=s+1}^{T} e_t^2\left(\mathbf{b}, \mathbf{d}, \mathbf{l}, a\right), \qquad (2)$$

where $e_t\left(\mathbf{b}, \mathbf{d}, \mathbf{l}, a\right)$ is defined as

$$e_t\left(\mathbf{b}, \mathbf{d}, \mathbf{l}, a\right) \equiv y_t - \mathbf{b}'\mathbf{x}_{t-s_x} - \mathbf{d}'\mathbf{x}_{t-s_x}\mathbf{I}\left(\mathbf{l}'\mathbf{q}_{t-s_q} > a\right),$$

and $\left(\mathbf{b}', \mathbf{d}', \mathbf{l}', a\right)'$ is a generic element of the parameter space of $\left(\boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\lambda}', \alpha\right)'$. Due to the jump discontinuity in the objective function induced by $\mathbf{I}(\cdot)$, the estimator for $\left(\boldsymbol{\lambda}', \alpha\right)'$ is super-consistent: it converges at a rate equal to $(T-s)$ and it is not asymptotically normally distributed, with asymptotic distribution that generally depends on a host of nuisance parameters. The least squares estimator $\left(\hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\delta}}'\right)'$ for $\left(\boldsymbol{\beta}', \boldsymbol{\delta}'\right)'$ is $(T-s)^{1/2}$ asymptotically normally distributed and asymptotically independent of that for $\left(\boldsymbol{\lambda}', \alpha\right)'$: inference on $\left(\boldsymbol{\beta}', \boldsymbol{\delta}'\right)'$ can be performed as if $\left(\boldsymbol{\lambda}', \alpha\right)'$ were known. See Chan (1993) and Hansen (2000) for technical details.

From a computational standpoint, we propose a two-step algorithm to estimate $\left(\boldsymbol{\lambda}', \alpha\right)'$ and $\left(\boldsymbol{\beta}', \boldsymbol{\delta}'\right)'$ in (1) by sequential minimization of the residual sum of squares: we first estimate $\left(\boldsymbol{\lambda}', \alpha\right)'$ by constructing an objective function defined on the parameter space of $\left(\boldsymbol{\lambda}', \alpha\right)'$ only; we then estimate $\left(\boldsymbol{\beta}', \boldsymbol{\delta}'\right)'$ given the estimator for $\left(\boldsymbol{\lambda}', \alpha\right)'$ obtained in the previous step. The proposed algorithm is implemented as follows:

1. Define the set $\mathbf{L} \equiv \mathbf{L}_1 \times \cdots \times \mathbf{L}_{k_q}$ so that $\mathbf{l}$ in (2) is such that $\mathbf{l} \equiv \left(l_1, \ldots, l_{k_q}\right)' \in \mathbf{L}$ and $l_j \in \mathbf{L}_j$ for $j = 1, \ldots, k_q$, where $\times$ denotes the Cartesian product operator: $\mathbf{L}$ can be constructed according to economic arguments such as those put forward in Section 5. For $\mathbf{l} \in \mathbf{L}$ define the set $\mathbf{A}(\mathbf{l})$ so that a in (2) is such that $a = a(\mathbf{l}) \in \mathbf{A}(\mathbf{l})$: following Tong and Lim (1980), the elements of $\mathbf{A}(\mathbf{l})$ are a subset of the quantiles of the empirical distribution function of $\mathbf{l}'\mathbf{q}_{t-s_q}$. Define

$$\mathbf{x}_{t-s_x}\left[\mathbf{l}, a(\mathbf{l})\right] \equiv \left\{\mathbf{x}_{t-s_x}', \mathbf{x}_{t-s_x}'\mathbf{I}\left[\mathbf{l}'\mathbf{q}_{t-s_q} > a(\mathbf{l})\right]\right\}'$$

and consider

$$\begin{aligned}
&\left\{\hat{\boldsymbol{\beta}}\left[\mathbf{l}, a(\mathbf{l})\right]', \hat{\boldsymbol{\delta}}\left[\mathbf{l}, a(\mathbf{l})\right]'\right\}' \\
&\quad = \left\{\sum_{t=s+1}^{T} \mathbf{x}_{t-s_x}\left[\mathbf{l}, a(\mathbf{l})\right]\mathbf{x}_{t-s_x}\left[\mathbf{l}, a(\mathbf{l})\right]'\right\}^{-1} \\
&\qquad \times \left\{\sum_{t=s+1}^{T} \mathbf{x}_{t-s_x}\left[\mathbf{l}, a(\mathbf{l})\right]y_t\right\};
\end{aligned}$$

[1] In particular, the constant c introduced in Theorem 3.1 can take any value on the real line: in this case, the result stated in Theorem 3.1 still holds, but the proof has to be suitably generalized.