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An efficient and fair solution for communication graph games

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ABSTRACT

We introduce an efficient solution for games with communication graph structures and show that it is characterized by efficiency, fairness and a new axiom called fair distribution of the surplus.

1. Introduction

A cooperative game with transferable utility, shortly TU-game, is a pair consisting of a finite set of players and a characteristic function assigning a worth to each coalition of players. Myerson (1977) introduced TU-games with restricted cooperation possibilities represented by an undirected graph which nodes represent the players and the edges *communication links* between the players. Under the assumption that only coalitions of connected players can cooperate, Myerson derived the so-called *restricted game* and proposed the Shapley value of this restricted game as a solution for such graph games. This so-called *Myerson value* is characterized by component efficiency and fairness.

Component efficiency states that for each component of the graph the total payoff to its players equals the worth of that component. However, when the characteristic function is not superadditive, component efficiency might be not desirable. Consider, for example, the graph game on $N = \{1, 2, 3\}$ with only players 1 and 2 connected and characteristic function v with worth 5 for coalition $\{1, 2\}$ as well as for the single player coalition $\{3\}$, and worth 8 for the grand coalition N. A solution satisfying component efficiency allocates 5 to players 1 and 2 together and also 5 to player 3, which is more than the worth of the grand coalition. But also under superadditivity efficiency might be required instead of component efficiency. For example, consider a research fund that has money available to distribute amongst individual researchers. Every researcher that submits a proposal takes part in the distribution of the budget. However, the fund has the policy to stimulate interdisciplinary research and therefore proposals submitted by two or more researchers receive relatively bigger amounts of money. Therefore, a subset of connected researchers can secure a bigger grant by submitting a joint proposal. Although usually not all researchers are connected to each other, still the full research budget is available and will be distributed. This requires a value to satisfy efficiency.

We introduce an efficient value for graph games that is characterized by efficiency and two other axioms, namely the Myerson's *fairness* axiom saying that deleting a link between two players affects both players' payoff equally, and a new axiom *fair distribution of the surplus* that compares for every component the total payoff to this component in the game itself to the total payoff of this component in the subgame induced by this component. For the research fund example it is obvious that the presence of joint proposals affects the size of the grant. The fair distribution of the surplus condition requires that these effects are balanced. The value for graph games characterized by the three axioms equals the Shapley value when the graph is complete and is equal to the equal surplus division when the graph is empty. Recently, Casajus (2007)

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also proposed an efficient value for graph games. We provide an example where Casajus's value favors standalone players, whereas our value favors cooperating players.

In the next section we introduce some basic definitions and notation. In Section 3 the new value is given and characterized. We also compare our solution with several others.

2. Preliminaries

A *TU-game* is a pair $\langle N, v\rangle$, where $N \subset \mathbb{N}$ is a finite set of at least two players and $v: 2^N \to \mathbb{R}$ a characteristic function on N with $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of S can obtain total payoff $v(S)$ by agreeing to cooperate. The set of all characteristic functions on N is denoted by $\mathcal{G}^N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$. For nonempty $T \subset N$, the *subgame* of $v \in \mathcal{G}^N$ with respect to T is $v_T \in \mathcal{G}^T$ defined as $v_T(S) = v(S)$, $S \subseteq T$. For $K \subset \mathbb{N}$, we denote $\mathbb{R}^K$ as the k-dimensional vector space whose elements $x \in \mathbb{R}^K$ have components x_i, $i \in K$. The cardinality of a set A is denoted by $|A|$ or the corresponding lower case letter $a = |A|$.

For $v \in \mathcal{G}^N$, a payoff vector $x \in \mathbb{R}^N$ assigns payoff x_i to $i \in N$. A single-valued solution, called *value*, is a mapping ξ that assigns to every $\langle N, v\rangle$ a payoff vector $\xi(v) \in \mathbb{R}^N$. A value ξ is *efficient* if $\sum_{i\in N} \xi_i(v) = v(N)$ for every $\langle N, v\rangle$. The best-known efficient value is the Shapley value (Shapley, 1953), given by

$$Sh_i(v) = \sum_{\{S\subseteq N \mid i\in S\}} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S\setminus\{i\})),$$

for all $i \in N$.

For $N \subset \mathbb{N}$, a communication structure on N is specified by a graph $\langle N, \Gamma\rangle$ with $\Gamma \subseteq \Gamma^N = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$, i.e., Γ is a collection of unordered pairs of players, where $\{i,j\}$ represents a communication *link* between players $i, j \in N$, and $\langle N, \Gamma^N\rangle$ is the complete graph on N. For simplicity of notation we often write Γ instead of $\langle N, \Gamma\rangle$ and we denote the set of all graphs on N by $\mathcal{L}^N$. A pair $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$ constitutes a *graph game* on N. For nonempty $T \subseteq N$, the subgraph $\Gamma \in \mathcal{L}^N$ with respect to set T is $\Gamma|_T \in \mathcal{L}^T$ defined by $\Gamma|_T = \{\{i,j\} \in \Gamma \mid i,j \in T\}$.

A sequence of different nodes $(i_1, \ldots, i_k)$, $k \geq 2$, is a path from i_1 to i_k in Γ, if for all $h = 1, \ldots, k-1$, $\{i_h, i_{h+1}\} \in \Gamma$. A graph $\Gamma \in \mathcal{L}^N$ is connected if there is a path in Γ between any two nodes. For $\Gamma \in \mathcal{L}^N$, coalition $S \subseteq N$ is connected, if the subgraph $\Gamma|_S$ is connected. Subset $T \subseteq S$ is a *component* of S if (i) $\Gamma|_T$ is connected, and (ii) for every $i \in S \setminus T$, subgraph $\Gamma|_{T\cup\{i\}}$ is not connected. We denote by N/Γ the set of all components of N. We also denote the set $S/\Gamma|_S$ of all components of S in the subgraph $\Gamma|_S$ shortly by S/Γ and the element of S/Γ containing $i \in S$ by $(S/\Gamma)_i$.

A *graph game value* is a mapping ξ that assigns a payoff vector $\xi(v,\Gamma) \in \mathbb{R}^N$ to every $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$. The best-known is the Myerson value. In Myerson (1977) it is assumed that only connected coalitions are able to cooperate. A non-connected coalition S can only realize the sum of the worths of its components in S/Γ. This yields the restricted game $v^\Gamma \in \mathcal{G}^N$ defined by

$$v^\Gamma(S) = \sum_{T\in S/\Gamma} v(T), \quad \text{for all } S \subseteq N.$$

The Myerson value μ assigns to every graph game $\langle v, \Gamma\rangle$ the Shapley value of v^Γ, so $\mu(v,\Gamma) = Sh_i(v^\Gamma)$. The Myerson value is the unique graph game value that satisfies component efficiency and Myerson's fairness. A graph game value ξ satisfies component efficiency if for every $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$, for every $C \in N/\Gamma$, $\sum_{i\in C} \xi_i(v,\Gamma) = v(C)$. As argued in the introduction, we want to have a value to be efficient and also satisfying fairness.

Efficiency. For every $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$, it holds $\sum_{i\in N} \xi_i(v,\Gamma) = v(N)$.

Fairness. For every $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$ and for every $\{i,j\} \in \Gamma$, it holds $\xi_i(v,\Gamma) - \xi_i(v,\Gamma_{-ij}) = \xi_j(v,\Gamma) - \xi_j(v,\Gamma_{-ij})$, where $\Gamma_{-ij} = \Gamma \setminus \{\{i,j\}\}$.

3. Efficiency, fairness and fair distribution of the surplus

We look for a graph game value that is characterized by efficiency, fairness and a new axiom that we refer to as fair distribution of the surplus.

Fair distribution of the surplus (FDS). For every graph game $\langle v, \Gamma\rangle$ on any player set N, and for any two components $C, C' \in N/\Gamma$, it holds

$$\frac{1}{c}\sum_{i\in C}(\xi_i(v,\Gamma) - \xi_i(v_C, \Gamma|_C)) = \frac{1}{c'}\sum_{i\in C'}(\xi_i(v,\Gamma) - \xi_i(v_{C'}, \Gamma|_{C'})).$$

Notice that $\langle v_C, \Gamma|_C\rangle$ and $\langle v_{C'}, \Gamma|_{C'}\rangle$ are defined on the reduced player sets C, respectively, C'. For a component $C \in N/\Gamma$, the axiom compares the payoffs that the players of C receive in the game itself to the payoffs that these players receive in the subgame on C. It states that the change in the average payoff of the players in a component C equals the change in the average payoff of the players in any other component C'. The FDS axiom only states a requirement when Γ has at least two components, otherwise the requirement reduces to an identity. Further, FDS is weaker than component efficiency. This follows straightforwardly since component efficiency implies that $\sum_{i\in C} \xi_i(v,\Gamma) = \sum_{i\in C} \xi_i(v_C, \Gamma|_C) = v(C)$, for all $C \in N/\Gamma$. Fair distribution of the surplus is equivalent to saying that for every component $C \in N/\Gamma$,

$$\frac{1}{c}\sum_{i\in C}(\xi_i(v,\Gamma) - \xi_i(v_C, \Gamma|_C)) = \frac{1}{n}\sum_{i\in N}\left(\xi_i(v,\Gamma) - \xi_i(v_{(N/\Gamma)_i}, \Gamma|_{(N/\Gamma)_i})\right).$$

We show that there is a unique graph game value that satisfies efficiency, fairness and FDS.

Theorem 3.1. *There is a unique graph game value satisfying efficiency, fairness and fair distribution of the surplus. This is the value ψ given by*

$$\psi(v,\Gamma) = Sh_i(\overline{v}^\Gamma) \quad \text{for all } \langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N \text{ and every } N \subset \mathbb{N},$$

where for a player set $N \subset \mathbb{N}$ and $\langle v, \Gamma\rangle \in \mathcal{G}^N \times \mathcal{L}^N$, the game $\overline{v}^\Gamma \in \mathcal{G}^N$ is given by

$$\overline{v}^\Gamma(S) = \begin{cases} v^\Gamma(S), & S \subsetneq N, \\ v(N), & S = N. \end{cases}$$

Note that the solution is obtained by taking the Shapley value of the Myerson restricted game v^Γ, except that $\overline{v}^\Gamma(N) = v(N)$ instead of $v^\Gamma(N) = \sum_{H\in N/\Gamma} v(H)$. Indeed the worth of the 'grand coalition' N in the restricted game should be $v(N)$ to get efficiency. It turns out that this modification is sufficient to obtain the unique graph game value satisfying efficiency, fairness and FDS.

Proof of Theorem 3.1. We first prove that ψ satisfies the three axioms. Since $\overline{v}^\Gamma(N) = v(N)$, efficiency follows by efficiency of the Shapley value.