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A simple test for the equality of integration orders

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HIGHLIGHTS

- We propose a test for the equality of integration orders.
- The test is valid in stationary/nonstationary, invertible/noninvertible circumstances.
- Unlike many other methods, our proposal is valid under cointegration.
- The test is simple to compute and it enjoys standard asymptotics.

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ABSTRACT

A necessary condition for two time series to be nontrivially cointegrated is the equality of their respective integration orders. Thus, it is standard practice to test for order homogeneity prior to testing for cointegration. Tests for the equality of integration orders are particular cases of more general tests of linear restrictions among memory parameters of different time series, for which asymptotic theory has been developed in parametric and semiparametric settings. However, most tests have been just developed in stationary and invertible settings, and, more importantly, many of them are invalid when the observables are cointegrated (because they involve inversion of an asymptotically singular matrix). We propose a general testing procedure which does not suffer from this serious drawback, and, in addition, it is very simple to compute, it covers the stationary/nonstationary and invertible/noninvertible ranges, and, as we show in a Monte Carlo experiment, it works well in finite samples.

1. Introduction

Recently, the concepts of fractionally integration and cointegration have raised the attention of numerous researchers. This new framework introduces additional challenges, because, in practice, those (possibly noninteger) integration orders are unknown, so the traditional way of testing for cointegration, based on ideas like the ones of Dickey and Fuller (1979) or Phillips and Perron (1988), needs to be revised. For example, given two observable series, $y_t, x_t, t = 1, \ldots, n$, a necessary condition for these processes to be nontrivially cointegrated (so a linear combination of them has a smaller order) is the equality of their respective integration orders. Thus, it is standard practice to test for order homogeneity prior to testing for cointegration. Tests for the equality of integration orders are particular cases of more general tests of linear restrictions among memory parameters of multivariate time series, which have been developed mainly assuming stationarity and invertibility. In the parametric setting rigorous asymptotic theory has been developed by Heyde and Gay (1993) and Hosoya (1997). In the semiparametric framework, under local assumptions, Wald tests of linear restrictions on memory parameters have been proposed for the stationary case by Robinson (1995a) and Lobato (1999), whereas, additionally, results in Robinson (1995b) suggest the use of Lagrange Multiplier and Likelihood Ratio tests (see Marinucci and Robinson, 2001). These semiparametric tests enjoy standard asymptotics (feature also shared by the parametric ones), but suffer from a serious drawback, because they are invalid in case there exists cointegration among the series. The reason is that the test statistics involve inversion of an asymptotically singular matrix. This problem was acknowledged by Marinucci and Robinson (2001), and Robinson and Yajima (2002) offered a sensible solution at cost of introducing an additional user-chosen number.

The present paper proposes a testing procedure for the equality of integration orders of two fractionally integrated processes. The test covers the stationary/nonstationary and invertible/noninvertible ranges and it is valid irrespective of whether the time series are cointegrated or not. In addition, its computation just requires estimation of integration orders and of the spectral density of the short memory input series which originate the fractionally integrated processes at frequency zero. Finally, assuming very mild conditions, our proposed test statistic

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enjoys standard asymptotics under the null hypothesis of equality of orders.

In the next section we present our testing procedure, which is rigorously justified in Appendix. Section 3 includes a Monte Carlo study of finite-sample behavior and, finally, we conclude in Section 4.

2. Testing for the equality of integration orders

Consider the bivariate process $z_t = (y_t, x_t)'$, prime denoting transposition, $t \in Z, Z = \{0, \pm 1, \ldots\}$, where

$$y_t = \Delta^{-\delta_y} \{v_{1t} 1 (t > 0)\}, \qquad y_t = 0, t \leq 0, \tag{1}$$

$$x_t = \Delta^{-\delta_x} \{v_{2t} 1 (t > 0)\}, \qquad x_t = 0, t \leq 0, \tag{2}$$

where $1(\cdot)$ is the indicator function, $\Delta = 1 - L$, L is the lag operator, and formally

$$\Delta^{-\alpha} = \sum_{j=0}^{\infty} a_j (-\alpha) L^j, \; a_j (\alpha) = \frac{\Gamma (j + \alpha)}{\Gamma (\alpha) \Gamma (j + 1)}, \quad \alpha \neq 0, -1, -2, \ldots,$$

where $\Gamma(\cdot)$ represents the gamma function, taking $\Gamma(\alpha) = \infty$ for $\alpha = 0, -1, -2, \ldots, \Gamma(0)/\Gamma(0) = 1$. We introduce

Assumption A. The process $v_t = (v_{1t}, v_{2t})'$, $t \in Z$, has representation

$$v_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} j \left\| A_j \right\|^2 < \infty, \tag{3}$$

with $\|\cdot\|$ denoting the Euclidean norm, where

(i) ε_t are independent and identically distributed vectors with mean zero, positive definite covariance matrix Ω, $E \|\varepsilon_t\|^q < \infty, q > 2$;
(ii) $f_{ii}(0) > 0$, $i = 1, 2$, where $f_{ij}(0)$ is the (i, j) element of the spectral density of v_t, denoted by $f(\lambda)$.

Assumption A seems satisfactorily mild, easily covering stationary and invertible autoregressive moving average systems. Under this assumption, model (1), (2) implies that y_t, x_t, are Type II fractionally integrated processes of orders δ_y, δ_x, respectively (see, e.g. Robinson, 2005).

We introduce additional notation. For any sequence s_t and any real c, let $s_t(c) = \Delta^c \{s_t 1 (t > 0)\}$, and related to y_t, x_t in (1), (2), for real c, d, define $z_t(c, d) = (y_t(c), x_t(d))'$. Finally, let "$\sim$" mean "exact rate of convergence".

Consider estimators $\widehat{\delta}_x, \widehat{\delta}_y, \widehat{f}(0)$ of $\delta_x, \delta_y, f(0)$ respectively, such that the following condition holds.

Assumption B. As $n \to \infty$,

$$\widehat{f}(0) \to_p f(0),$$

and for $\kappa > 0$ and $K < \infty$,

$$\widehat{\delta}_x - \delta_x \sim n^{-\kappa}, \qquad \widehat{\delta}_y - \delta_y \sim n^{-\kappa},$$

where

$$\left|\widehat{\delta}_x\right| + \left|\widehat{\delta}_y\right| \leq K. \tag{4}$$

Assumption B, although unprimitive, is very mild. (4) is innocuous if $\widehat{\delta}_x, \widehat{\delta}_y$, optimize over compact sets. If we assume a parametric structure for v_t, $\sqrt{n}$-consistent estimators of the orders of integration and $f(0)$ are achievable by a multivariate extension of the results in Robinson (2005), which extended the results in Velasco and Robinson (2000) to cover our type of processes. This rate is far better than needed, so we might be content by assuming some weak conditions of smoothness of the spectral density of v_t around frequency zero, and estimate the orders and $f(0)$ semiparametrically. In particular, the estimators of Robinson (1995a,b), justified by Robinson (2005) for our type of processes, satisfy Assumption B. Also, given estimators $\widehat{\delta}_x$, $\widehat{\delta}_y$, the nonparametric estimator of $f(0)$ could be based on weighted averages of the periodogram of the proxy $\widehat{v}_t = (y_t(\widehat{\delta}_y), x_t(\widehat{\delta}_x))'$ of v_t.

We introduce our test statistic. Let $h_n > 0$ be a sequence (whose role will be clarified in Remark 2 below) such that

$$h_n^{-1} + n^{-\kappa} h_n \to 0 \quad \text{as } n \to \infty. \tag{5}$$

Define

$$\widehat{a} = \left(1\left(n^{\kappa}(\widehat{\delta}_y - \widehat{\delta}_x) > h_n\right), 1\left(n^{\kappa}(\widehat{\delta}_y - \widehat{\delta}_x) \leq h_n\right)\right)', \tag{6}$$

and let

$$\widehat{t} = \frac{(2\pi n)^{-1/2} \widehat{a}' \sum_{t=1}^{n} z_t \left(\widehat{\delta}_x, \widehat{\delta}_y\right)}{\left(\widehat{a}' \widehat{f}(0) \widehat{a}\right)^{1/2}}, \tag{7}$$

be the test statistic for $H_0 : \delta_y = \delta_x$ against the alternative $H_1 : \delta_y \neq \delta_x$.

Theorem 1. *Let (1), (2) and Assumptions A and B hold. Then*

$$\widehat{t} \to_d N(0, 1) \quad \text{under } H_0; \qquad \widehat{t} \sim n^{|\delta_x - \delta_y|} \quad \text{under } H_1. \tag{8}$$

Remark 1. As shown in the proof of Theorem 1, $\widehat{a} \to_p a \equiv (1, 0)' 1\left(\delta_y > \delta_x\right) + (0, 1)' 1\left(\delta_y \leq \delta_x\right)$, so $\widehat{t}$ is asymptotically equivalent to $(2\pi n)^{-1/2} \sum_{t=1}^{n} y_t(\widehat{\delta}_x)/\widehat{f}_{11}^{1/2}(0)$, or, alternatively, to $(2\pi n)^{-1/2} \sum_{t=1}^{n} x_t(\widehat{\delta}_y)/\widehat{f}_{22}^{1/2}(0)$, depending on whether $\delta_y > \delta_x$ or $\delta_y \leq \delta_x$, respectively. Thus, asymptotically, $\widehat{t}$ is based on underdifferenced processes under H_1 (which is precisely the source of power), whereas under H_0 it is based on $x_t(\widehat{\delta}_y)$ (although it could have been equally based on $y_t(\widehat{\delta}_x)$ with a slight modification of the definition of $\widehat{a}$).

Remark 2. It would have been natural to set $h_n = 0$ in (6). In this case, the test would have been based on $y_t(\widehat{\delta}_x)$ if $\widehat{\delta}_y > \widehat{\delta}_x$, or $x_t(\widehat{\delta}_y)$ if $\widehat{\delta}_y \leq \widehat{\delta}_x$. However setting $h_n = 0$ in (6) implies that under H_0 the limit of $\widehat{a}$ is random. This leads to a very complicated limit dependence between the numerator and denominator of (7), so the simple and neat result (8) would no longer hold.

Remark 3. If y_t and x_t were cointegrated, $f(0)$ would be singular. This is precisely the reason why the different semiparametric tests considered in the literature are not valid with cointegration, as they require inversion of a matrix which tends in probability to a singular matrix (usually the equivalent to $f(0)$ in a more general framework). As can be inferred from Remark 1, singularity of $f(0)$ does not affect our test procedure as long as $f_{ii}(0) > 0$, $i = 1, 2$.

Remark 4. Although we just consider Type II processes, this was just motivated by the uniform treatment of any value of δ_x and δ_y that this definition allows, all results holding equally for corresponding Type I processes (see Robinson, 2005).

Remark 5. We do not consider deterministic components in (1), (2). However, there is no loss of generality here, because these components can be eliminated by taking appropriate integer differences of the observables and then applying our procedure to the zero-mean differenced series. Taking integer differences might lead to differenced processes with negative integration orders, which, in particular, might fall in the noninvertibility region. This complicates the likability of Assumption B, although procedures like Hualde and Robinson (2011) or Hurvich and Chen (2000) are appropriate in these circumstances.