Economics Letters 120 (2013) 262–266

Contents lists available at SciVerse ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Envelope condition method versus endogenous grid method for solving dynamic programming problems

Lilia Maliar *, Serguei Maliar

Hoover Institution, Stanford University, USA
University of Alicante, Spain

HIGHLIGHTS

- We introduce the envelope condition method (ECM) for solving dynamic programming problems.
- ECM simplifies rootfinding and is faster than conventional value function iteration.
- ECM is similar in accuracy and speed to Carroll's (2005) endogenous grid method (EGM).
- We introduce accurate EGM and ECM that approximate derivatives of value function.
- Codes are available.

ARTICLE INFO

Article history:
Received 5 January 2013
Received in revised form
14 April 2013
Accepted 19 April 2013
Available online 24 April 2013

JEL classification:
C6
C61
C63
C68

Keywords:
Numerical dynamic programming
Value function iteration
Endogenous grid
Envelope condition
Curse of dimensionality
Large scale

ABSTRACT

We introduce an envelope condition method (ECM) for solving dynamic programming problems. The ECM method is simple to implement, dominates conventional value function iteration and is comparable in accuracy and cost to Carroll's (2005) endogenous grid method. Codes are available.

1. Introduction

Dynamic programming methods are an important tool in economics; see Judd (1998), Santos (1999), Rust (2008) and Stachursky (2009) for reviews. Conventional value function iteration (VFI) goes backward: we guess a value function in period $t+1$, and we solve for a value function in period t using the Bellman equation. Conventional VFI is expensive: it requires us to find a root to a non-linear equation in all grid points, which involves interpolating value function off the grid and approximating conditional expectation in a large number of candidate solution points; see Aruoba et al. (2006) for examples assessing the cost of VFI.

Carroll (2005) introduces an endogenous grid method (EGM) that simplifies rootfinding under time iteration. The idea is to construct a grid on future endogenous state variables instead of current endogenous state variables, which are treated as unknowns. In a typical economic model, it is easier to solve for current endogenous state variables given the future state variables than to solve for future endogenous state variables given the current state variables. This is why EGM dominates conventional VFI.

In this paper, we have two contributions. First, we introduce *an envelope condition method (ECM)*, another solution method that

* Correspondence to: T24, Hoover Institution, 434 Galvez Mall, Stanford University, Stanford, CA 94305-6010, USA. Tel.: +1 650 725 3416.
E-mail address: maliarl@stanford.edu (L. Maliar).

0165-1765/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.econlet.2013.04.031

simplifies rootfinding in dynamic programming problems. ECM does not perform conventional backward iteration on the Bellman equation but iterates forward. Also, to construct policy functions, ECM uses the envelope condition instead of the first-order conditions used by conventional VFI and EGM. We find that systems of equations produced by ECM are typically easier to solve than those produced by conventional VFI. In this sense, ECM is similar to EGM.

Second, we introduce versions of the EGM and ECM methods that approximate derivatives of value function instead of value function itself. We find that these versions produce far more accurate solutions than do similar methods that approximate value function itself.

We compare the EGM and ECM methods using both analytical arguments and numerical examples. We find that EGM and ECM are nearly identical in terms of accuracy and speed in our test problem, the neoclassical growth model with elastic labor supply. Codes are available at http://www.stanford.edu/~maliarl.

2. The model

We study the standard neoclassical growth model with elastic labor supply.

2.1. Bellman equation

We solve for value function V that satisfies the Bellman equation,

$$V(k,a)=\max_{k',c,\ell}\left\{u(c,\ell)+\beta E\left[V\left(k',a'\right)\right]\right\} \tag{1}$$

$$\text{s.t. } k'=(1-\delta)k+af(k,\ell)-c, \tag{2}$$

$$\ln a'=\rho\ln a+\epsilon',\quad \epsilon'\sim\mathcal{N}\left(0,\sigma^2\right), \tag{3}$$

where k, c, ℓ and a are capital, consumption, labor and productivity level, respectively; $\beta\in(0,1)$; $\delta\in(0,1]$; $\rho\in(-1,1)$; $\sigma\geq 0$; the utility and production functions, u and f, respectively, are strictly increasing, continuously differentiable and concave; the primes on variables denote next-period values, and $E\left[V\left(k',a'\right)\right]$ is an expectation conditional on state (k,a).

2.2. Optimality conditions

We divide the optimality conditions in two blocks. The first block identifies policy functions that correspond to a given value function V, and the second block identifies a value function that corresponds to given policy functions.

2.2.1. Block 1: identifying policy functions given a value function

If a solution to Bellman equation (1)–(3) is interior, the optimal quantities satisfy first-order conditions (FOCs) with respect to labor and consumption and the envelope condition, which, respectively, are

$$u_\ell(c,\ell)=-u_c(c,\ell)\,af_\ell(k,\ell), \tag{4}$$

$$u_c(c,\ell)=\beta E\left[V_k\left(k',a'\right)\right], \tag{5}$$

$$V_k(k,a)=u_c(c,\ell)\left[1-\delta+af_k(k,\ell)\right], \tag{6}$$

as well as budget constraint (2). Here, $F_x(\ldots,x,\ldots)$ denotes a first-order partial derivative of function $F(\ldots,x,\ldots)$ with respect to variable x.

2.2.2. Block 2: identifying a value function given policy functions

In the optimum, value function V and its derivative V_k satisfy

$$V(k,a)=u(c,\ell)+\beta E\left[V\left(k',a'\right)\right], \tag{7}$$

$$V_k(k,a)=\beta\left[1-\delta+af_k(k,\ell)\right]E\left[V_k\left(k',a'\right)\right]. \tag{8}$$

Condition (7) is Bellman equation (1) evaluated under the optimal policy functions (which makes a maximization sign unnecessary), and condition (8) follows by combining (5) and (6).

2.3. Discussion

Envelope condition (6) is central to our analysis.[1] Observe that we have two conditions that describe the relation between V_k and the policy functions: one is FOC (5) and the other is envelope condition (6). Conventional VFI and EGM of Carroll (2005) approximate policy functions using FOC (5), namely, they solve the system (2), (4) and (5). In contrast, our ECM method will approximate policy functions using envelope condition (6), namely, it will solve the system (2), (4) and (6). In Sections 3 and 4, we show that the system of equations built on envelope condition (6) is easier to solve than the system of equations built on conventional FOC (5), in which case ECM is a preferred choice.

Furthermore, the envelope condition provides a basis for condition (8). This condition allows us to approximate V_k without finding V. Under our construction, all methods described in the paper can approximate a solution by iterating on either (7) or (8) or both, whereas the previous literature including conventional VFI and EGM of Carroll (2005) iterate only on Bellman equation (7). In Section 5, we show that the iteration on (8) leads to far more accurate solutions than iteration on (7).

3. The model with inelastic labor supply

We first consider a model with inelastic labor supply under the following assumptions

$$u(c,\ell)=\frac{c^{1-\gamma}-1}{1-\gamma}\quad\text{and}\quad f(k,\ell)=k^\alpha, \tag{9}$$

where $\gamma>0$ and $\alpha\in(0,1)$. In this case, we have $\ell=1$, and FOC (4) is absent.

3.1. Conventional VFI

The conventional VFI method makes a guess on the future value function $V\left(k',a'\right)$ and identifies policy functions using budget constraint (2) and FOC (5). By substituting c from (2) into (5) under the assumptions (9), we obtain

$$\beta E\left[V_k\left(k',a'\right)\right]=\left[k'-(1-\delta)k-ak^\alpha\right]^{-\gamma}. \tag{10}$$

We must solve (10) for k' in each grid point (k,a). Finding a solution to (10) is expensive. For example, if we parameterize V with a polynomial function, then solving (10) includes interpolation of V_k to new values $\left(k',a'\right)$, as well as approximation of conditional expectation $E\left[V_k\left(k',a'\right)\right]$. We must explore many different candidate values of $\left(k',a'\right)$ until we find a solution to (10).

3.2. Endogenous grid method

The EGM of Carroll (2005) also makes a guess on the future value function $V\left(k',a'\right)$ and identifies policy functions using budget constraint (2) and FOC (5). The difference is that EGM treats the future endogenous state variable as fixed, and it treats the current endogenous state variable as unknown. Since the values for k' are fixed, it is possible to compute up-front $E\left[V\left(k',a'\right)\right]\equiv W\left(k',a\right)$ and $E\left[V_k\left(k',a'\right)\right]\equiv W_k\left(k',a\right)$.

[1] Typically, the envelope condition is used to derive the Euler equation (namely, (6) is updated to get $V_k\left(k',a'\right)$ and the result is substituted into (5) to eliminate the unknown derivative of the value function). In the present paper, we do not derive the Euler equation but concentrate on the envelope condition in the form (6).