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Large distributional games with traits[☆]

M. Ali Khan [a,*], Kali P. Rath [b], Haomiao Yu [c], Yongchao Zhang [d]

[a] Department of Economics, The Johns Hopkins University, Baltimore, MD 21218, USA
[b] Department of Economics, University of Notre Dame, Notre Dame, IN 46556, USA
[c] Department of Economics, Ryerson University, Toronto, ON M5B2K3, Canada
[d] School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China

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ABSTRACT

A comprehensive theory of large strategic games with (socioeconomic and biological) traits (*LSGT*) has recently been presented in Khan et al. (2012, 2013), and we present a reformulation pertaining to large distributional games with traits (*LDGT*).

1. Introduction

The theory of large non-atomic games in strategic form has by now gone well beyond a framework in which a player's choice among a finite number of actions depends on a statistical summary, be it an average or a distribution, of the plays of everyone else in the game; see Khan and Sun (2002). Recent work of Khan et al. (2012, 2013) has offered a rich but analytically tractable framework for game-theoretic situations in which the payoffs of the non-denumerable number of players depend not only on the actions chosen by other players but also on underlying characteristics of the players themselves, be they socioeconomic or biological.[1] The resulting theory does not conflate *names* and *traits*, and with these separated out, goes beyond the existence of pure-strategy equilibria to obtain comprehensive results on the continuity, asymptotic implementation and *ex post* characterizations of such equilibria. However, the theory jettisons the Lebesgue interval as the space of player-names in favor of so-called saturated measure spaces; indeed it offers a characterization of such spaces.

It is well-known that Mas-Colell (1984) proposed the jettisoning of the space of player-names altogether, and presented a "reformalization... in terms of distributions rather than measurable functions". He emphasized the simplicity of his conception.

> We shall see how once the definitions are available we get a (pure strategy) equilibrium existence theorem quite effortlessly and under general conditions . . . Because of (insubstantial) measurability problems an *a priori* given representation of the game is thus a sort of straightjacket. The approach via distributions frees one from it. This accounts for its comparative ease.

Indeed, if one invokes the identity map and sees a player's name also as his characteristic, it is now well-understood that in a setting with finite actions, Mas-Colell's theorem is an almost trivial consequence of that of Schmeidler (1973), and it is only

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* Corresponding author. Tel.: +1 410 516 8545; fax: +1 410 516 7600.
E-mail address: akhan@jhu.edu (M.A. Khan).

[1] These considerations have been emphasized in Akerlof and Kranton (2000, 2002) and Brock and Durlauf (2001); see Blume et al. (2010) and their references.

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in more general contexts that the process of *symmetrization*, as opposed to *purification*, can be exploited, and connections with the "marriage lemma" and "disintegrations" made, resulting in the distributional approach coming into its own.[2] In any case, the variant has attracted both theoretical and applied interest, and the two approaches, one based on a random variable and the other on its law, are simply complementary ways of looking at a large game, two sides of the same coin.

With this view of the antecedent background literature, the following questions naturally suggest themselves:

(i) What shape does the theory of large strategic games with traits (henceforth *LSGT*), with its informationally-richer notion of agent interdependence, take when recast for large distributional games with traits (henceforth *LDGT*)?
(ii) What role does the notion of a saturated probability space play in this reformulated theory?

We answer each of these questions here in a way that has the satisfying feature that it recovers previous results simply by the specialization of the separable metric space of traits to a singleton! And to be sure, writing two and half decades after Mas-Colell, we can rely on previous advances and bring to bear heavier mathematical artillery to resolve questions that he did not address. All in all, the results bear testimony not only to the simplicity but the fecundity of Mas-Colell's conception.

This letter is then organized as follows. Section 1 introduces saturated spaces and uses them to record an antecedent result on *LSGT*, Proposition 1. Section 2 turns to *LDGT* and presents the necessary definitions of the game, the Nash and symmetric Nash equilibrium distributions of the game (*NED* and symmetric *NED*), the realization of these distributions and their closed graph property. Section 3 presents the results. Theorems 1 and 2 generalize the corresponding results on *NED* in Mas-Colell (1984) and Khan and Sun (1995b). The other results represent the extended reach of the theory: Theorem 3 is on the closed graph property of a NED correspondence; Proposition 2, Theorem 4 and Corollary 1 are results on Lebesgue and saturated realizations of *NED*, symmetric and non-symmetric. The proofs of these results are all in keeping with Mas-Colell's original conception — they are natural and effortless consequences of Keisler and Sun (2009) and Khan et al. (2013) once the necessary definitions are in place.

2. Large strategic games with traits (*LSGT*)

The principal motivation behind the development of *LSGT* is the need for a rich space of player characteristics, embodied in a universal space of traits T, a complete separable metrizable (Polish) space, and a particular distribution ρ on the induced Borel σ-algebra $\mathcal{B}(T)$ on this space. The resulting "externality" notion then embraces both distributions on T and the common action set A assumed to be compact metric. This involves $\mathcal{M}(T \times A)$, the space of probability distributions on the Borel product σ-algebra on $T \times A$, and its subspace $\mathcal{M}^{\rho}(T \times A)$ of distributions whose marginal distribution on T is ρ. The space of players' payoffs $\mathcal{V}_{(A,T,\rho)}$ is then given by the space of all continuous functions on the product space $A \times \mathcal{M}^{\rho}(T \times A)$, and endowed with its resulting Borel σ-algebra induced by the sup-norm topology. It is the measurable space $(\mathcal{V}_{(A,T,\rho)}, \mathcal{B}(\mathcal{V}_{(A,T,\rho)}))$. The space of players' names $(I, \mathcal{I}, \lambda)$, an atomless probability space, thus does not figure as part of the players' characteristics. We can now present a formulation of a *LSGT* as follows.

[2] For these claims, see the Handbook chapter cited here as Khan and Sun (2002), and other papers of Khan, Rath and Sun, therein. This chapter also cites Khan and Sun (1995a) as the conceptual predecessor to the notion of preferences depending on distributions of consumptions across finite observable characteristics or types.

Definition 1. A *LSGT* is a measurable function $\mathcal{G}$ from I to $T \times \mathcal{V}_{(A,T,\rho)}$ such that $\lambda\mathcal{G}_1^{-1} = \rho$, where $\mathcal{G}_k$ is the projection of $\mathcal{G}$ on its kth-coordinate, $k = 1, 2$. A *(pure strategy) Nash equilibrium* of $\mathcal{G}$ is a measurable function $f : I \to A$, such that for λ-almost all $i \in I$, and with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \to T$ abbreviated for $\mathcal{G}_1$,

$$v_i\left(f(i), \lambda(\alpha, f)^{-1}\right) \geq v_i\left(a, \lambda(\alpha, f)^{-1}\right) \quad \text{for all } a \in A.$$

In order to develop a general result on the existence of a Nash equilibria in *LSGT*, we need the notion of a saturated space as a formalization of the space of player-names.

Definition 2. A probability space is said to be countably-generated if its σ-algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not countably-generated. A probability space $(I, \mathcal{I}, \lambda)$ is saturated if it is nowhere countably-generated, in the sense that, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not countably-generated, where $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.

From the above definition, it is clear that saturated probability spaces must be atomless. An important property of saturated spaces is with respect to the saturation property defined below and encapsulated in a fact due to Hoover–Keisler and available in Keisler and Sun (2009).

Definition 3. An atomless probability space $(I, \mathcal{I}, \lambda)$ is said to have the saturation property for a Borel probability measure ν on the product of Polish spaces $X \times Y$ if for every measurable mapping $f: I \to X$ which induces the distribution of the marginal measure of ν over X, then there is a measurable mapping $g: I \to Y$ such that the induced distribution of the pair (f, g) on $(I, \mathcal{I}, \lambda)$ is ν.

Fact 1. *A probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if it has the saturation property for every Borel probability measure ν on the product of any two Polish spaces.*

The following antecedent result is from Khan et al. (2013, Theorem 1).

Proposition 1. *Every LSGT $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ has a Nash equilibrium if either of the following two (sufficient) conditions hold:* (i) *T and A are both countable spaces, and* (ii) *$(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

3. Large distributional games with traits (*LDGT*)

We now turn to the formulation where the space of player-names is suppressed, and rather than measurable functions, the focus shifts to their distributions.[3]

Definition 4. A *LDGT* is a distribution μ^{ρ} on $T \times \mathcal{V}_{(A,T,\rho)}$ such that μ^{ρ}_T, the marginal of μ^{ρ} on T, is ρ. A *NED* of μ^{ρ} is a distribution on τ on $T \times \mathcal{V}_{(A,T,\rho)} \times A$ if the marginal of τ on $T \times \mathcal{V}_{(A,T,\rho)}$ is μ^{ρ}, and if $\tau(B_\tau) = 1$ where

$$B_\tau = \{(t, v, a) \in (T \times \mathcal{V}_{(A,T,\rho)} \times A) : v(a, \tau_{T\times A}) \geq v(x, \tau_{T\times A}) \text{ for all } x \in A\},$$

and $\tau_{T\times A}$ is the marginal of τ on $T \times A$.

[3] In previous work, this resulting game form *LDGT* was referred to as a "large anonymous game", and its equilibrium distribution referred to as a *CNED*.