Economics Letters 117 (2012) 263–267

Contents lists available at SciVerse ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Fairness and fairness for neighbors: The difference between the Myerson value and component-wise egalitarian solutions☆

Sylvain Béal[a,*], Eric Rémila[b], Philippe Solal[c]

[a] Université de Franche-Comté, CRESE, 30 Avenue de l'Observatoire, 25009 Besançon, France
[b] Université de Lyon, LIP, UMR 5668 CNRS-ENS Lyon-Université Lyon 1, France
[c] Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France

ARTICLE INFO

Article history:
Received 22 February 2012
Received in revised form
5 May 2012
Accepted 18 May 2012
Available online 26 May 2012

JEL classification:
C71

Keywords:
Myerson value
Component-wise egalitarian solutions
Fairness
Fairness for neighbors
Two-step value

ABSTRACT

We replace the axiom of fairness used in the characterization of the Myerson value (Myerson, 1977) by fairness for neighbors in order to characterize the component-wise egalitarian solution. When a link is broken, fairness states that the two players incident to the link should be affected similarly while fairness for neighbors states that a player incident to the link and any of his other neighbors should be affected similarly. Fairness for neighbors is also used to characterize the component-wise egalitarian surplus solution and a two-step egalitarian surplus solution. These results highlight that egalitarian and marginalistic allocation rules can be obtained by applying the same equal gain/loss property to different types of players.

1. Introduction

Cooperative games with transferable utility describe situations in which any subset of the player set is able to form as a coalition and to earn the corresponding worth. However, in many situations the set of feasible coalitions is restricted by some hierarchical, technical or communicational structure. In this note, we consider communication situations, which consist of a cooperative game and an undirected graph modeling the limited communication structure. The vertices in the graph represent the players and the edges represent the communication links between the players. One of the most famous allocation rules for communication situations is the Myerson value (Myerson, 1977), which is the Shapley value (Shapley, 1953) of the so-called Myerson restricted game. Myerson (1977) characterizes the Myerson value by component efficiency and fairness. Component efficiency means that the sum of payoffs in any component of the graph equals the worth of the component. Fairness means that the deletion of a communication link between two players hurts or benefits both players equally.

We keep component efficiency as an axiom and the equal gain/loss principle used in fairness. However, rather than requiring equal payoff variations between a player and the neighbor incident to the deleted link, we require equal payoff variations between this player and each of his other neighbors. So, the resulting axiom, which we call fairness for neighbors, relies on the same principle as fairness. The only difference is that the payoff variations involve those neighbors of the players incident to the deleted link that are neglected by the axiom of fairness. In other words, fairness points out the role of the neighbor with which a player is no longer linked while fairness for neighbors points out the role of the neighbors with which a player continues to be linked. The rational behind this property can be understood as follows. When the removal of a link breaks a component into two parts, the players incident to that link do not communicate anymore. Nevertheless, both players still communicate with their other neighbors, and so are able to cooperate with the members of the corresponding component. Therefore, it makes sense to apply the equal gain/loss principle to pairs of players who continue to communicate in the resulting graph. It turns out that replacing fairness by fairness for

☆ The authors want to thank an anonymous referee for helpful comments. Financial support by the National Agency for Research (ANR) —research program "Models of Influence and Network Theory" (MINT) ANR.09.BLANC-0321.03— and the "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD) project is gratefully acknowledged.

* Corresponding author. Tel.: +33 3 81 66 68 26; fax: +33 3 81 66 65 76.
E-mail addresses: sylvain.beal@univ-fcomte.fr (S. Béal), eric.remila@ens-lyon.fr (E. Rémila), solal@univ-st-etienne.fr (P. Solal).

0165-1765/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.econlet.2012.05.020

neighbors and adding an equal treatment principle in two-player communication situations yield the component-wise egalitarian solution, which distributes the worth of each component of the graph equally among its members.

Replacing the equal treatment principle in two-player communication situations by the classical axiom of standardness characterizes the component-wise egalitarian surplus solution, which first assigns to every player his own worth, and then distributes the remaining surplus of his component equally among its members. Standardness requires that the allocation rule coincides with the standard solution in two-player components. Since the Myerson value coincides with the component-wise egalitarian surplus solution in two-player communication situations, both allocations rules are comparable in the sense that they essentially differ with respect to pairs of players to whom the equal gain/loss principle applies when a communication link is severed. Somehow, the difference between an egalitarian rule and a marginalistic rule is only explained by the type of neighbors with which a player's payoff variation is evaluated.

We also define and characterize a two-step procedure in which the equal surplus sharing principle is applied between components and then within each component so as to construct an efficient component-wise egalitarian surplus solution. This approach is inspired by the two-step Shapley value introduced by Kamijo (2009) for TU-games with a coalition structure. The resulting allocation rule satisfies fairness for neighbors and is characterized by replacing component efficiency in the characterization of the component-wise egalitarian surplus solution by efficiency, covariance and a natural axiom requiring that two components of the graph obtain the same total payoff if they enjoy the same worth.

This note therefore provides axiomatic characterizations of three egalitarian allocation rules for communication situations that are comparable with the Myerson value in the sense that they incorporate a similar equal gain/loss principle and satisfy other properties. This research is related to the work of Slikker (2007) who provides comparable axiomatizations of the Myerson value and of the component-wise egalitarian solution for cooperative network games. Our characterization of the component-wise egalitarian solution can be considered as closer to the characterization of the Myerson value since the equal gain/loss property, which is the corner stone of the axiom of fairness, is reused in the axiom of fairness for neighbors, even if it is applied to different pairs of players. The present article is also related to the work of van den Brink (2007) who shows that replacing null players by nullifying players in the characterization of the Shapley value characterizes the egalitarian solution for TU-games. A player is nullifying if its presence in a coalition generates zero worth.

The rest of the note is organized as follows. Section 2 provides preliminaries. The component-wise egalitarian solution and the component-wise egalitarian surplus solution are characterized in Sections 3 and 4, respectively. In Section 5, we characterize the two-step component-wise egalitarian surplus solution. Section 6 concludes by a comparison table.

2. Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of players. A *cooperative game with transferable utility* on N, or simply a TU-game, is a *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each $S \in 2^N$, $v(S)$ is the *worth* of *coalition* S and s its cardinality. A TU-game v is *zero-normalized* if $v(\{i\}) = 0$ for each $i \in N$. For any TU-game v, any real $a \in \mathbb{R} \setminus \{0\}$ and any $b \in \mathbb{R}^n$, the TU-game $(av + b)$ is such that, for each $S \in 2^N$, $(av + b)(S) = av(S) + \sum_{i \in S} b_i$. The zero-normalization of a game v is thus obtained by setting $a = 1$ and $b_i = -v(\{i\})$ for each $i \in N$.

A *communication graph* is a pair (N, L) where the nodes in N represent the players and edges in $L \subseteq \{\{i, j\} \subseteq N : i \neq j\}$ represent bilateral communication links between players. We shall use the short notations ij instead of $\{i, j\}$ and $L \setminus ij$ instead of $L \setminus \{\{i, j\}\}$. For each player $i \in N$, $L_i = \{j \in N : ij \in L\}$ is the set of *neighbors* of i in (N, L). For each coalition S, $L(S) = \{ij \in L : i \in S, j \in S\}$ is the set of links between players in S. The graph $(S, L(S))$ is the *subgraph* of (N, L) induced by S. A sequence of $p \geq 1$ distinct players $(i_1, \dots, i_p)$ is a *path* in (N, L) if $i_q i_{q+1} \in L$ for $q = 1, \dots, p - 1$. A graph (N, L) is *connected* if there exists a path between any two players in N. A coalition S is connected in (N, L) if $(S, L(S))$ is a connected graph. A coalition C is a *component* of a graph (N, L) if the subgraph $(C, L(C))$ is connected and for each $i \in N \setminus C$, the subgraph $(C \cup \{i\}, L(C \cup \{i\}))$ is not connected. Let N/L and $S/L(S)$ be the sets of components of (N, L) and $(S, L(S))$, respectively.

A *communication situation* on N is a pair (v, L) such that v is a TU-game on N and (N, L) a communication graph. Denote by $\mathcal{C}_N$ the set of all communication situations on N. A *payoff vector* $x \in \mathbb{R}^n$ is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. An *allocation rule* on $\mathcal{C}_N$ is a function f that assigns to each $(v, L) \in \mathcal{C}_N$ a payoff vector $f(v, L) \in \mathbb{R}^n$. Given a communication situation (v, L), the *graph-restricted game* v^L, introduced by Myerson (1977), is defined as

$$\forall S \in 2^N, \quad v^L(S) = \sum_{T \in S/L(S)} v(T).$$

If a coalition is not connected, then its worth in v^L is given by the sum of the worths of its connected components. The Shapley value (Shapley, 1953) of v^L is known as the *Myerson value* of (v, L). More specifically, the Myerson value is the allocation rule μ on $\mathcal{C}_N$ defined as

$$\forall (v, L) \in \mathcal{C}_N, \forall i \in N,$$
$$\mu_i(v, L) = \sum_{S \in 2^N : i \in S} \frac{(n-s)!(s-1)!}{n!} \big(v^L(S) - v^L(S \setminus \{i\})\big).$$

Myerson (1977) characterizes μ by component efficiency and fairness. Component efficiency requires that an allocation rule assigns to any component C of (N, L) the total payoff $v(C)$. Fairness states that for every communication link in the graph the incident players lose or gain the same amount from breaking this link.

Component efficiency. For each $(v, L) \in \mathcal{C}_N$ and each $C \in N/L$, it holds that

$$\sum_{i \in C} f_i(v, L) = v(C).$$

Fairness. For each $(v, L) \in \mathcal{C}_N$ and each $ij \in L$, it holds that

$$f_i(v, L) - f_i(v, L \setminus ij) = f_j(v, L) - f_j(v, L \setminus ij).$$

3. The component-wise egalitarian solution

The *component-wise egalitarian solution* is the allocation rule CE on $\mathcal{C}_N$ that distributes the worth of each component equally among its members in any communication situation, i.e

$$\forall (v, L) \in \mathcal{C}_N, \forall C \in N/L, \forall i \in C, \quad CE_i(v, L) = \frac{v(C)}{c}.$$

The component-wise egalitarian solution satisfies component efficiency but not fairness. It does satisfy the following equal gain/loss property.

Fairness for neighbors. For each $(v, L) \in \mathcal{C}_N$, each $ij \in L$, and each $k \in L_i \setminus \{j\}$, it holds that

$$f_i(v, L) - f_i\big(v, L \setminus ij\big) = f_k(v, L) - f_k\big(v, L \setminus ij\big).$$