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An alternative explanation for the logit form probabilistic choice model from the equal likelihood hypothesis

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ABSTRACT

We present an alternative explanation of the logit probabilistic choice from the equal likelihood hypothesis without the Gumbel distribution. The hypothesis is that if the total utility values from combinations of actions are the same, all such combinations of actions are equally likely.

1. Introduction

The logit choice model, first proposed by Luce (1959), has been frequently used in the probabilistic choice problem, including quantal response equilibrium (referred to as QRE, (McKelvey and Palfrey, 1995)); see Train (2003). The logit probability $P(u_j)$ that an agent chooses the action with utility u_j is given by

$$\Pr(u_j) = \frac{\exp(\lambda u_j)}{\int \exp(\lambda u_i) di}. \tag{1}$$

The justification of this function form of the probabilistic choice problem is usually understood in the way that actual utility u_j^A is divided into two factors: observed utility u_j and a random term ε_j. We have

$$u_j^A = u_j + \varepsilon_j, \tag{2}$$

where the ε_j are independent and identically distributed (i.i.d.) and follow the Gumbel distribution; see Gumbel (2004). The Gumbel distribution is characterized by the cumulative distribution function $F(\varepsilon) = e^{-e^{-\varepsilon}}$, which is a distribution of extreme values. The probability of taking a certain action then follows the logit function in Eq. (1), which was shown by McFadden (1972). The problem with this justification is that the random noise follows the Gumbel distribution. If it were Gaussian, it would be persuasive. While we find that the logit form probabilistic choice function is effective, some researchers do not prefer to use it. In this paper, we present an alternative explanation of the basis of the logit form probabilistic choice function without assuming the Gumbel random term. The mathematical structure of our derivation is the entropy maximization principle. It is already known that the logit function is derived from the entropy maximization principle; see, Wilson (1967), Anas (1983), Soofi (1992), and Aoki and Yoshikawa (2007). However, sufficient meaning and justification of entropy maximization principle in view of the individual action principle is not given enough. In some sense, the contribution of the present paper is providing the meaning and the justification. It is an equal likelihood hypothesis. This hypothesis is that if the values of the total utility (U) from combinations of actions are the same, then all such combinations of actions are equally likely. We will show that, if the hypothesis is true, then the probability that an agent chooses an action follows the logit function. In addition it is reinterpreted in the following way. If a person chooses an action with some probability in order to satisfy the equal likelihood hypothesis, the probability is a logit function. Anderson et al. (1988) derived the logit form demand function by specifying the form of utility function which directly had the entropy form function, such as $x_i \log(x_i)$. However, that the utility function had the form of an

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entropy function which leads to a logit function was assumed from the beginning, and this assumption played a central role. They did not treat it as a probabilistic choice problem but as a deterministic demand function of the representative consumer. On the other hand, we do not assume any specific form of utility function. Instead, we assume an equal likelihood hypothesis, which will be explained below. Although we do not know which approach is better, the logit function is so central that it is useful to have further underpinnings for it and to understand it from another aspect.

2. Explanation from the equal likelihood hypothesis

Suppose that there are k types of action, $a_1, a_2, \ldots, a_k$, from which an agent derives utilities $u_1, u_2, \ldots, u_k$; an agent chooses actions from $\{a_1, a_2, \ldots, a_k\}$ for a total of N times. Later on we will find that N is arbitrary. Suppose that an agent has bounded rationality, so that an agent does not maximize the utility; hence, it is a stochastic choice problem. Let n_j denote the number of action a_j taken by an agent, where j runs from 1 to k such that $n_1 + n_2 + \cdots + n_k = N$, and let U denote the total utility of the agent from all the N actions, which is given by

$$U \equiv n_1u_1 + n_2u_2 + \cdots + n_ku_k. \tag{3}$$

We now define the combination of actions. For example, the combination of actions 12123 means that an agent takes action a_1 the first time, a_2 the second time, a_1 the third time, a_2 the fourth time, and a_3 the fifth time. We then define the vector of the number of actions: $\vec{n} \equiv (n_1, n_2, \ldots, n_k)$. This denotes that an agent takes action a_1 n_1 times, action a_2 n_2 times, ..., and action a_k n_k times. In the above example, the agent takes action a_1 twice, a_2 twice, and a_3 once, which is denoted by $(n_1, n_2, n_3) = (2, 2, 1)$. Although the vector of the number of actions is the same, there are many combinations of actions satisfying the vector of the number of actions. In the above example, besides 12123, other combinations of actions such as 11223, 22311, 23211, 31212, and so on, satisfy the same vector of the number of actions $(n_1, n_2, n_3) = (2, 2, 1)$.

We are considering a probabilistic choice problem. To derive such probabilities, we follow a hypothesis. It is natural to assume that, if combinations of actions give the same total utility U, then an agent takes such combinations of actions with the same probability. We then introduce the following hypothesis.

Hypothesis 1 (*Equal Likelihood Hypothesis*)**.** If the values of the total utility U of combinations of actions $(n_1, n_2, \ldots, n_k)$ are the same, all such combinations of actions are equally likely.

Example 1. Suppose that an agent chooses actions from the set of actions $\{a_1, a_2, a_3\}$ $N = 100$ times and that the utilities associated with the actions are given by $\{u_1, u_2, u_3\} = \{2, 2, 1\}$. The equal likelihood hypothesis states that an agent takes the combinations of actions with the following vectors of the number of actions $(n_1, n_2, n_3) = (40, 40, 20)$, $(45, 35, 20)$, and $(50, 30, 20)$ with the same probability because all the combinations give the same total utility, $U = 180$. Other vectors of the number of actions which are equally likely exist. This hypothesis does not specify the total utility of the combination of actions taken by an agent. It only states that the combinations of actions with the same total utility U are equally likely. The hypothesis does not say anything about the probabilities of combinations of actions $(n_1, \ldots, n_k)$ with different total utilities. Although it seems natural that it is more likely that an agent chooses a combination of actions with higher total utility, the hypothesis does not say anything about this. We do not have to assume anything about the probabilities of different total utilities to derive the logit form function. The hypothesis should be tested by an experiment or an empirical study. It cannot be derived from a system, because it is a hypothesis. What we can state now is that assuming this hypothesis is natural for the stochastic choice problem, and the logit form function is derived from the hypothesis.

The number of combinations of actions satisfying the vector of the number of actions $(n_1, n_2, \ldots, n_k)$ is given by

$$\frac{N!}{n_1!n_2!\cdots n_k!}. \tag{4}$$

Example 2. We now explain how the equal likelihood hypothesis gives the probability of a combination of actions with an example. Suppose that an agent chooses actions from $\{a_1, a_2, a_3, a_4\}$ $N = 4$ times, and that the associated utilities are given by $\{u_1, u_2, u_3\} = \{0, 1, 2, 3\}$. The combinations of actions satisfying any of the following vectors of the number of actions $(n_1, n_2, n_3, n_4) = (2, 0, 1, 1), (1, 1, 2, 0), (1, 2, 0, 1)$, and $(0, 3, 1, 0)$ give the same total utility, $U = 5$. They are all the possible combinations of actions. The number of combinations of actions satisfying the above four vectors of the number of actions are respectively given by $\frac{4!}{2!} = 12$, $\frac{4!}{2!} = 12$, $\frac{4!}{2!} = 12$, and $\frac{4!}{3!} = 4$. The equal likelihood hypothesis states that all of these $12 \cdot 3 + 4 = 40$ combinations of actions are equally likely, because all of them give the same total utility, $U = 5$. Therefore, the combinations of actions satisfying $(n_1, n_2, n_3, n_4) = (2, 0, 1, 1)$ occur with probability $\frac{12}{40}$, $(1, 2, 0, 1)$ with $\frac{12}{40}$, , $(1, 2, 0, 1)$ with $\frac{12}{40}$, and $(0, 3, 1, 0)$ in $\frac{4}{40}$.

2.1. Deriving the logit function from the equal likelihood hypothesis

What follows is the main proposition of the present paper.

Proposition 1. *If the equal likelihood hypothesis is satisfied and the numbers of actions n_i satisfy $n_i \gg 1$ for all i, then the probability $\Pr(u_j)$ that an agent chooses the action a_j with utility u_j follows the logit function:*

$$\Pr(u_j) = \frac{\exp(\lambda u_j)}{\int \exp(\lambda u_i)di} \quad \forall j. \tag{5}$$

Proposition 1 is also reinterpreted in the following way.

If a person chooses an action in order to satisfy the equal likelihood hypothesis when the number of actions n_j is large for all j, then that person chooses the action with logit probability $\Pr(u_j)$.

Before we prove Proposition 1, we present the following formula.

Theorem 1 (*Stirling's Formula*)**.** $\log(x!) = x(\log x - 1)$ *as* $x \to \infty$.

Proof. See Arfken et al. (1995) or Lebedev and Silverman (1972). □

Proof of Proposition 1. Let p_j denote the probability that an agent chooses the action a_j with utility u_j. The probabilities p_j and $\Pr(u_j)$ are the same, namely $\Pr(u_j) = p_j = n_j/N$. From the equal likelihood hypothesis, the most likely vector of the probabilities of actions $(p_1, p_2, \ldots, p_k)$ among those with the same total utility U maximizes the number of the combinations of actions, which is given by $N!/(n_1!n_2!\cdots n_k!)$, such that the following constraints are satisfied:

$$\sum_{j=1}^{k} n_j u_j = U, \qquad \sum_{j=1}^{k} n_j = N. \tag{6}$$