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"Hyperbolic" discounting: A recursive formulation and an application to economic growth

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ABSTRACT

We axiomatize decreasing impatience (DI) in a discrete-time setting, as originally discussed by Prelec, and formulate a class of recursively-defined discounting functions that conform to DI. The recursive formulation is used to analyze the Ramsey growth problem using dynamic programming techniques.

1. Introduction

Prelec (2004) axiomatizes a notion on preferences called decreasing impatience (DI) which he then relates to the shape of the discounting function. A simple prospect is a dated pair (x,t) where x is an outcome in period t and $\succcurlyeq$ represents preferences over prospects. Given that $(x,t)\sim(y,s)$ where $y>x>0$ and where $\sim$ denotes indifference; the agent is said to have decreasing impatience if for $t<s$ and any $\sigma>0$, we have $(y,s+\sigma)\succcurlyeq(x,t+\sigma)$. This relation captures the notion that an agent would be willing to wait for the larger reward y over the reward x if the time frame σ for making the comparison between y and x is pushed farther away in time. If preferences are represented in a time-separable discounted utility form, Prelec (2004) shows that DI is equivalent to convexity of the log of the discounting function. He defines a comparative notion of DI and shows that greater DI corresponds to greater (Pratt – Arrow) convexity of the log of the discounting function.

Although Prelec (2004) views DI as the "core property" expressed in the hyperbolic discounting literature, the latter has tended to focus on the case of quasi-hyperbolic discounting. For example, Ramsey optimal growth problems have been analyzed in a quasi-hyperbolic setting by Krusell and Smith (2003) and Judd (2005).[1] Such functions are viewed as a "useful simplification"[2] since they can be expressed recursively and easily incorporated in a standard dynamic programming framework; despite empirical and experimental evidence in favor of decreasing impatience.[3]

In Section 2, we express the axioms for DI and the comparative notion of DI in a discrete-time setting and present sufficient conditions on the discounting function that satisfy the axioms; we then discuss a recursive formulation of a discounting function that exhibits DI. We use this recursive formulation to analyze the Ramsey growth problem using a dynamic programming approach (Section 3) and compare the resulting generalized Euler equation to the case of quasi-hyperbolic $(\beta-\delta)$ discounting (Section 4).

2. Decreasing impatience — A recursive formulation

Define a sequence $\{\beta_t\}_{t=0}^{+\infty}$ where $\beta_t\in R$ such that $\beta_t=\Pi_{j=0}^{t}r_j$. The sequence $\{r_t\}_{t=0}^{+\infty}$ is such that $r_0=1$. The following ensures that the

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[1] The seminal paper by Phelps and Pollak (1968) introduces hyperbolic discounting in an intergenerational setting. Laibson (1994, 1997) and Harris and Laibson (2001) analyze implications of $\beta-\delta$ discounting in an intrapersonal setting. Young (2007) analyzes the case of a finite, multi-period, deviation from quasi-hyperbolic discounting.

[2] Prelec (2004), pp. 512.

[3] See for example Halevy (2007).

sequence $\{\beta_t\}_{t=0}^{+\infty}$ is a discounting function, meaning that $\beta_0=1$ and $\beta_t\to 0$ as $t\to+\infty$.

Assumption 0. $r_j\in(0,1)$ and $r_j\to r_*<1$.

Proposition 1. Under Assumption 0, $\{\beta_t\}_{t=0}^{+\infty}$ is a discounting function.[4]

Assumption 0 is not necessary for $\{\beta_t\}_{t=0}^{+\infty}$ to be a discounting function; for example, if $r_{2j}=1/2$ and $r_{2j+1}=1$ for all $j>0$ then $\{\beta_t\}_{t=0}^{+\infty}$ is a discounting function. From now on, we assume that the sequence $\{r_j\}_j$ is such that $\{\beta_t\}_{t=0}^{+\infty}$ is a discounting function.

We follow Prelec (2004) and let $\{(x_i,t_i):i\in N\}$ represent a *temporal prospect* (shorthand, $(x,t)_i$), a sequence of consumption streams x_i in a particular period t_i for $i\in N$. A *simple prospect* is a sequence with one element denoted (x,t). We assume that preferences $\succcurlyeq$ are represented by a payoff function $u(\cdot)$ and a discounting function $\{\beta_t\}_{t=0}^{+\infty}$.[5] The period$-t$ self of this agent weakly prefers prospect $(x,t)_i$ to $(y,t)_i$, i.e. $(x,t)_i\succcurlyeq(y,t)_i$, if and only if $\sum_i\beta_{t_i}u(x_i)\geq\sum_i\beta_{t_i}u(y_i)$. Prelec defines decreasing impatience for simple prospects, expressed here in a discrete-time setting.

Definition 1. Prelec (2004) $\succcurlyeq$ exhibits decreasing impatience (or DI) if for any $s,t\in\{0,1,2,\cdots\}$ and any $y>x>0$ such that $(x,t)\sim(y,s)$ then $(y,s+i)\succcurlyeq(x,t+i)$ for all $i\in\{1,2,\cdots\}$ (with $(y,s+i)\succ(x,t+i)$ for strict DI).

While it is clear that exponential and quasi-hyperbolic discounting satisfy the discrete-time notion of DI expressed in this definition, it can be shown that they do not satisfy strict DI. Examples satisfying strict DI include the hyperbolic discounting functions[6] $\beta_t=(1+\alpha t)^{-\beta/\alpha}$ and $\beta_t=\beta^{t^\alpha}$ with $\alpha\neq 1$. Neither of the last two discount factors, however, can be expressed as a time–invariant function of the discount factor from the last period. Given the popularity of recursive formulations and techniques in dynamic optimization problems, this rules out the analysis of interesting problems where the economic agent acts as though her discounting function exhibits decreasing impatience. In this paper we propose a family of discounting functions that can be expressed recursively and that meet the discrete-time version of DI. We turn our attention to characterizing DI in a discrete-time setting.

Let $\succcurlyeq$ be a preference order represented by utility function $u(x)$, and discount function $\{\beta_t\}_{t=0}^{+\infty}$. The next proposition shows that $\succcurlyeq$ exhibits (strict) DI if the sequence $\{r_j\}_j$ is (strictly) increasing.

Assumption 1. $r_{j+1}\geq r_j$ for all $j>0$.

Assumption 1'. $r_{j+1}>r_j$ for all $j>0$.

Proposition 2. Under Assumption 1 (1'), the preference ordering $\succcurlyeq$ exhibits (strict) decreasing impatience.

In a continuous-time setting, Prelec shows that $\succcurlyeq$ exhibits DI if and only if $ln\beta_t$ is convex in t.[7] We provide a local property on the discounting function that guarantees DI in a discrete-time setting. Notice that (strict) convexity of $ln\beta_t$ implies in particular that $\ln\beta_t<(\leq)\frac{1}{2}\ln\beta_{t-1}+\frac{1}{2}\ln\beta_{t+1}$. The latter inequality holds if and only if Assumption 1 (1') holds.

Example 1. Consider the following class of recursively-defined discounting functions.

$$\begin{aligned}\beta_{t+1}&=r_{t+1}\beta_t \text{ for } t=0,1,\cdots\\ r_{t+1}&=\alpha r_t+\gamma(1-\alpha) \text{ for } t=1,\cdots\end{aligned}\tag{1}$$

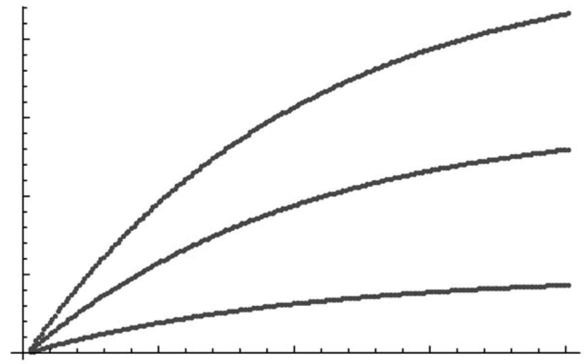

Fig. 1. *x*-axis: time. y-axis: r_t.

Where $\alpha,\gamma\in(0,1)$ and $\beta_0=1,r_0=1,r_1:=\psi<\gamma$. The sequence $\{r_j\}_j$ converges to $r^*=\gamma<1$ and $\{\beta_t\}_t$ is therefore a discounting function (Proposition 1). Note that the sequence $\{r_j\}_j$ is an increasing function if and only if $\alpha r_t+\gamma(1-\alpha)>r_t$ or $r_t<\gamma$ so one must choose $r_1=\psi$ where $\psi<\gamma$. One can express β_t as a function of α,γ,ψ and time t. First note that:

$$r_{t+1}=\alpha^t\psi+(1-\alpha)\gamma\sum_{i=0}^{t-1}\alpha^i \text{ for } t=0,1,\cdots.$$

Using Eq. (1) and simplifying, one obtains:

$$\beta_t=\prod_{j=0}^{t}\left[\alpha^j\psi+(1-\alpha)^j\gamma\right] \text{ for } t=0,1,\cdots.$$

Fig. 1 shows an example of the sequence $\{r_j\}$ where $\beta_1=r_1=0.2$, $\alpha=.99$ and $\gamma=0.3, 0.5$, and 0.7, respectively. The case where $\gamma=0.3$ is the bottom curve in both Figs. 1 and 2.

Let $\succcurlyeq^*$ be another preference order represented by the utility function $u^*(x)$, and the discounting function $\{\beta_t^*\}_{t=0}^{+\infty}$. We now define a comparative criterion for DI, inspired from Prelec's criterion.[8]

Definition 2. $\succcurlyeq$ exhibits more decreasing impatience than $\succcurlyeq^*$ if for any $0\leq t<s$ and outcomes $0<x<y$, $0<x'<y'$ such that $(x,t)\sim(y,s)$ and $(x',t)\sim^*(y',s)$ then for any $i>0$ and $k>0$ such that $(x,t+i)\succ(y,s+i+k)$ we have that $(x',t+i)\succ^*(y',s+i+k)$.

Label the agent whose preferences are represented by $\succcurlyeq$ ($\succcurlyeq^*$), agent 1 (2); suppose both of their preferences exhibit decreasing impatience. If $(x,t)\sim(y,s)$ then agent 1 will prefer the more distant prospect $(y,s+i)$ to $(x,t+i)$ for $i>0$ because of decreasing impatience. The definition states that if agent 1 experiences a “preference reversal” for some $k>0$ then it must be that this reversal has occurred for agent 2. We now state a sufficient condition for this to be true.

Assumption 2. $\frac{r^*_{s+1}}{r^*_s}<\frac{r_{s+1}}{r_s}$ for all $s>0$.

Proposition 3. Under Assumptions 1 and 2, the preference ordering $\succcurlyeq$ exhibits more decreasing impatience than $\succcurlyeq^*$.

In a continuous-time setting, Prelec shows that greater DI is characterized by a greater degree of log-convexity of the discounting function using the Pratt – Arrow criterion: if β_t and β_t^* are twice differentiable discounting functions, Prelec (2004) shows that $\succcurlyeq$ exhibits more DI than $\succcurlyeq^*$ if and only if [9]

$$\frac{d^2\ln\beta_t}{dt^2}\Big/\frac{d\ln\beta_t}{dt}\leq\frac{d^2\ln\beta_t^*}{dt^2}\Big/\frac{d\ln\beta_t^*}{dt}.\tag{2}$$

[4] All proofs are relegate to Appendix A, available online.

[5] We assume throughout this section that time begins in period 0. $u(\cdot)$ is assumed to be strictly increasing— this assumption is sufficient in a discrete-time setting.

[6] Hyperbolic discounting was first proposed by George Ainslie (1975), and generalized by Herrnstein (1981), Prelec (1989) and Loewenstein and Prelec (1992).

[7] Corollary to Proposition 1, pp. 520 of Prelec (2004).

[8] Prelec's comparative definition for DI relies on the continuity of time and cannot be used directly.

[9] See proof of Proposition 2, Prelec (2004), pp. 530.