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# Conceptual frameworks and experimental design in simultaneous equations ☆

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**Abstract**

Using examples drawn from two important papers in the recent literature on weak instruments, we demonstrate how observed experimental outcomes can be profoundly influenced by the different conceptual frameworks underlying two experimental designs commonly employed when simulating simultaneous equations.


*Keywords:* Simultaneous equations; Experimental design; Simulation experiment

*JEL classification:* C12; C15; C30

## 1. Simultaneous equations and experimental design

The classical linear simultaneous equations model has enjoyed renewed interest of late as a consequence of the problems associated with inference in weakly-identified models; see, for example, the papers discussed by Chesher et al. (2007). Several authors have addressed the specific problem of inference on the coefficient of an endogenous regressor in a structural equation and various suggestions have been made about how to proceed in such circumstances. Using invariance principles and similar regions, Andrews et al. (2006) restrict attention to a class of tests from which they extract members with desirable optimality properties. However, in the absence of a uniformly most powerful test, comparison of the absolute and relative merits of different tests continues to be guided by simulation experiments.

One aspect of this analysis that has received little attention is the observation that there are two formulations of the underlying model in common use. These formulations are probabilistically equivalent, in that one can move from one to the other via non-singular linear transformations. However, these transformations involve parameters that may be of interest in certain types of simulation experiments and, in such cases, the competing formulations of the models can be conceptually quite different. In this note we demonstrate that the lessons one might hope to learn through the use of simulation studies can vary enormously depending upon which conceptual framework is chosen to guide the experiments. We illustrate this point by exploring the behaviour of three statistics that have been important in the analysis of weakly-identified simultaneous equations models, namely the AR test (Anderson and Rubin, 1949), the K test (Kleibergen, 2002), and the conditional likelihood ratio (CLR) test (Moreira, 2003).

To begin, consider the classical structural equation model

$$y = Y\beta + X\gamma + u, \tag{1}$$

where the endogenous matrix variables $y$ and $Y$ are $N \times 1$ and $N \times n$, respectively, the matrix of exogenous variables $X$ is $N \times k$, and $u$ denotes an $N \times 1$ vector of uncorrelated stochastic disturbances with zero mean and variance $\sigma_u^2$. The vectors of structural coefficients $\beta$ and $\gamma$ are $n \times 1$ and $k \times 1$, respectively.

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There are two commonly encountered ways of completing the specification of this model. The first involves augmenting (1) by reduced form equations for $Y$; namely,

$$Y = X\Pi_1 + Z\Pi_2 + V, \tag{2}$$

with $\Pi_1$ and $\Pi_2$ of dimension $k\times n$ and $v\times n$, respectively. The stochastic specification is then completed by assumptions about the conditional distribution of $[u,V]$ given $X$ and $Z$, e.g. $[u,V]|[X,Z]\sim N(0,\Phi\otimes I_N)$, where

$$\Phi = \begin{bmatrix} \sigma_u^2 & \sigma_{uV} \\ \sigma_{Vu} & \Omega \end{bmatrix}.$$

That is, the rows of the $N\times(n+1)$ matrix $[u,V]$ are uncorrelated random vectors with zero mean and common $(n+1)\times(n+1)$ covariance matrix $\Phi$. Henceforth, the couplet of Eqs. (1) and (2), together with the accompanying distributional assumption, will be referred to as the structural equation specification (SES).

The second specifies a reduced form for all of the endogenous variables in the system; namely

$$[y,Y] = [X,Z]\begin{bmatrix} \pi_1 & \Pi_1 \\ \pi_2 & \Pi_2 \end{bmatrix} + [v,V], \tag{3}$$

where $[v,V]|[X,Z]\sim N(0,\Sigma\otimes I_N)$ with

$$\Sigma = \begin{bmatrix} \sigma_v^2 & \sigma_{vV} \\ \sigma_{Vv} & \Omega \end{bmatrix}, \tag{4}$$

and the coefficient vectors $\pi_1$ and $\pi_2$ are $k\times 1$ and $v\times 1$, respectively. Hereafter Eqs. (1) and (3), together with their distributional assumption, will be referred to as the reduced form specification (RFS). This model is comprised of more equations than there are endogenous variables and so compatibility of Eqs. (1) and (3) requires the parameter restrictions

$$\pi_1 - \Pi_1\beta = \gamma, \qquad \pi_2 = \Pi_2\beta, \qquad \sigma_u^2 = [1,-\beta']\Sigma[1,-\beta']', \tag{5}$$

which, together with Eq. (2), imply that

$$\Sigma = \begin{bmatrix} 1 & \beta' \\ 0 & I_n \end{bmatrix}\Phi\begin{bmatrix} 1 & 0' \\ \beta & I_n \end{bmatrix}. \tag{6}$$

Eqs. (5) and (6) show how the parameters of the two formulations of the model are related to each other. They also make clear that, in simulation experiments where values of $\beta$ are varied, it is impossible to simultaneously keep fixed the remaining parameters of both formulations. Hence, contingent upon which formulation of the model you prefer, the other becomes something of a moving feast as $\beta$ is varied, making comparison of simulation experiments across such paradigms very difficult. When investigating power, for example, the difficulty lies in presenting the power as a univariate function of $\beta$ when in fact the power curve sits on a multidimensional manifold. Making a choice between the two formulations of the model implies that one is traversing this manifold and passing through observationally equivalent parameter points in very different ways.

In order to illustrate the point, let us consider the problem of testing $H_0{:}\beta=0$ against the two-sided alternative $H_1{:}\beta\neq 0$ under the following two experimental designs based upon SES and RFS.

$$\begin{gathered} y = Y\beta + u \\ Y = Z\Pi_2 + V \\ [u,V]\sim N(0,\Phi\otimes I_N) \\ \Phi = \begin{bmatrix} 1 & \rho_{uV} \\ \rho_{uV} & 1 \end{bmatrix} \end{gathered}$$

Experimental Design 1 (ED1)

$$\begin{gathered} y = Z\Pi_2\beta + v \\ Y = Z\Pi_2 + V \\ [u,V]\sim N(0,\Phi\otimes I_N) \\ \Sigma = \begin{bmatrix} 1 & \rho_{vV} \\ \rho_{vV} & 1 \end{bmatrix} \end{gathered}$$

Experimental Design 2 (ED2)

Note from Eq. (6) that, under $H_0$, $\Sigma=\Phi$ and the two data generating mechanisms are observationally equivalent for any fixed values of the nuisance parameters. But the latter is not true in general. From Eq. (6) it is apparent that, as $\beta$ varies under $H_1$, one can choose to hold fixed either $\Sigma$ or $\Phi$, but not both. Hence it will be immaterial whether simulation experiments designed to investigate size properties are based on ED1 or ED2, but if one allows $\beta$ to vary under $H_1$, as one does when considering the power of certain tests, then the implications of using ED1 or ED2 can be very different.

Figs. 1 and 2 present power functions for the AR, K and CLR tests based on experimental designs ED1 and ED2, together with PE, the asymptotically efficient two-sided power envelope for invariant, similar tests, as described in Andrews et al. (2006). For both designs we have $n=1$, $N=100$ and $v=5$. To obtain Fig. 1 we have set $\lambda=\Pi_2'Z'Z\Pi_2=1.0$, so the instruments are weak. Fig. 1(a) is based on ED1 with $\rho_{uV}=0.99$, following Kleibergen (2002, Figure 4). Fig. 1(b) examines the effect of such weak instruments in ED2 where, instead of fixing $\rho_{uV}$, we set $\rho_{vV}=0.99$. In Fig. 2 we have set $\lambda=5.0$, so the instruments are stronger than for Fig. 1. Fig. 2(b), as in Andrews et al. (2006, Figure 1(a)), is based on ED2 with $\rho_{vV}=0.95$, so that the degree of endogeneity is weaker than in Fig. 1. Fig. 2(a) examines the corresponding power curves under ED1, where now $\rho_{uV}=0.95$.

The figures clearly demonstrate that the choice of experimental design has a profound effect upon the observed power characteristics of the different tests and hence a substantial influence on any conclusions that are likely to be