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A note on the accuracy of Markov-chain approximations to highly persistent AR(1) processes

Martin Flodén *

Stockholm School of Economics, Sweden
CEPR, United Kingdom

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Abstract

This note examines the accuracy of methods that approximate AR(1) processes with discrete Markov chains. Tauchen and Hussey's [Tauchen, G., Hussey, R., 1991. Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models. Econometrica 59, 371–396] method has problems under high autocorrelation. I suggest an alternative weighting function, and note that Tauchen's [Tauchen, G., 1986. Finite state Markov-chain approximations to univariate and vector autoregressions. Economics Letters 20, 177–181] method is relatively robust.

Keywords: Numerical methods; Income processes; Autoregressive process

JEL classification: C60

1. The problem

This note considers the accuracy of different methods that are commonly used to approximate autoregressive processes by Markov chains. We thus want to approximate the AR(1) process

$$z_{t+1} = (1-\rho)\mu + \rho z_t + \varepsilon_{t+1}$$

by an n-node Markov chain $\{Z,\Pi\}$ where $Z=\{z^1,z^2,\ldots,z^n\}$ and $\Pi=\{\pi_{i,j}\}$ where $\pi_{i,j}$ is the transition probability from z^i to z^j. Here $\varepsilon \sim N(0,{\sigma_\varepsilon}^2)$, and consequently the unconditional standard deviation of z is $\sigma_z=\sigma_\varepsilon(1-\rho^2)^{-1/2}$.

Five alternative methods are here used to approximate this AR(1) process. The first method follows Tauchen (1986). The nodes Z are then equally spaced between $\pm 1.2\sigma_z \ln n$, and the transition probabilities Π are the probabilities $\pi_{i,j}=Pr(z' \in [z^j-s,z^j+s] | z=z_i)$ implied by the AR(1) process. The step size s is half the distance between nodes, i.e. $s=(z^2-z^1)/2$, except at the extreme nodes where the intervals are $(-\infty,z^1+s]$ and $[z^n-s,\infty)$.

The following three methods all build on Tauchen and Hussey's (1991). The nodes $\{z^i\}$ are then the Gaussian nodes for some normal distribution $N(\mu,\hat{\sigma}^2)$, and the difference between the approximation methods I use is the choice of $\hat{\sigma}^2$.[1]

To summarize Tauchen and Hussey's method, let $\{z^i\}$ and $\{w^i\}$ denote the Gaussian quadrature nodes and weights for the normal distribution $N(\mu,\hat{\sigma}^2)$. Suppose now that $z_t=\mu$ and that ${\sigma_\varepsilon}^2=\hat{\sigma}^2$. Then these Gaussian nodes and weights typically provide an excellent approximation of how z will develop in the next period. The problem is that if $z_t\neq\mu$, Gaussian quadrature would imply other nodes (and weights), but the Markov chain requires that the nodes are fixed. So, can we find nodes Z and probabilities Π that provide an approximation to the process for any $z_t \in Z$? Gaussian quadrature provides nodes $\{z^i\}$ and weights $\{w^i\}$ so that

$$\int g(\xi)f(\xi)\mathrm{d}\xi \approx \sum g(z^i)w^i$$

* Department of Economics, Stockholm School of Economics, Box 6501, SE-11383 Stockholm, Sweden. Tel.: +46 8 7369263; fax: +46 8 313207.
E-mail address: martin.floden@hhs.se.

[1] Different choices for the mean μ could also have been considered.

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where g is some function, $\xi \sim N\,(\mu, \hat{\sigma}^2)$, and f is the density function for ξ. Tauchen and Hussey (1991) note that if

$$z_{t+1} = (1-\rho)\mu + \rho z_t + \xi_{t+1}$$

then

$$\int g(z_{t+1}) f(z_{t+1}|z_t)\mathrm{d}z_{t+1} = \int g(z_{t+1}) \frac{f(z_{t+1}|z_t)}{f(z_{t+1}|z_t=\mu)} f(z_{t+1}|z_t=\mu)\mathrm{d}z_{t+1},$$

i.e.

$$\int g(z_{t+1}) f(z_{t+1}|z_t)\mathrm{d}z_{t+1} \approx \sum g(z^i) \frac{f(z_{t+1}|z_t)}{f(z_{t+1}|z_t=\mu)} w^i.$$

They therefore propose that the AR(1) process should be approximated by the nodes $\{z^i\}$ and the transition probabilities

$$\pi_{i,j} = \frac{f(z^j|z^i)}{f(z^j|z_t=\mu)} w^i.$$

As already mentioned, I consider three variants of the Tauchen–Hussey method. First, as suggested by Tauchen and Hussey (1991) I use $\hat{\sigma}^2 = \sigma_\varepsilon^2$. Most subsequent implementations of the Tauchen and Hussey method (e.g. Burnside, 1999) use this alternative, and Tauchen and Hussey (their Section 3.3) argue that numerical evaluations support this choice.

The second variant I consider is $\hat{\sigma} = \sigma_z$. Tauchen and Hussey (1991) also mention this as a possible choice, and Klein (2007) suggests this specification. The third variant I consider is $\hat{\sigma} = w\sigma_\varepsilon + (1-w)\sigma_z$ where $w = 1/2 + \rho/4$. The variance $\hat{\sigma}^2$ is then set to a weighted average of the conditional and unconditional variances, and more weight is given to the conditional variance when the process is highly persistent.

Finally, the fifth method considered is outlined in Adda and Cooper (2003). This method first chooses n intervals such that z has equal unconditional probability to fall in each of the intervals. Second, one node is chosen for each interval, and this node is set to the expected value of z conditional on z being in that interval. Third, the transition probabilities are calculated as with Tauchen's (1986) method.

Using these five approximation methods, I consider the accuracy of approximations to different specifications of the AR(1) process

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}.$$

The Markov-chain approximations are often used in economics to model income processes, and I first consider the accuracy of approximations to three processes that have often been adopted in the literature. The first specification follows Aiyagari (1994) and sets $\rho=0.60$ and $\sigma_\varepsilon^2=0.013$, while the second follows Hubbard et al. (1995, HSZ) and sets $\rho=0.95$ and $\sigma_\varepsilon^2=0.030$. The final specification, suggested by Storesletten et al. (2000), is even more persistent and sets $\rho=0.98$ and $\sigma_\varepsilon^2=0.020$. In addition to examining these processes, I also examine the methods' accuracy for a broader set of autocorrelations.

2. Results

Table 1 reports the autocorrelation, conditional standard deviation, and unconditional standard deviation implied by the Markov chains that are obtained with the different approximation methods with n=5, 9, and 15 nodes, and Table 2 reports the

Table 1
Approximated AR(1) processes

	True	n=5					n=9					n=15				
		Tauch.	Tauchen–Hussey			A–C	Tauch.	Tauchen–Hussey			A–C	Tauch.	Tauchen–Hussey			A–C
			σ_ε	σ_z	w			σ_ε	σ_z	w			σ_ε	σ_z	w	
Aiyagari's process, ρ=0.60, σ_ε^2=0.013																
ρ	0.6000	0.5844	0.5992	0.6024	0.6000	0.5682	0.5982	0.6000	0.6000	0.6000	0.5938	0.5998	0.6000	0.6000	0.6000	0.5996
σ_ε	0.1140	0.1167	0.1137	0.1138	0.1139	0.1127	0.1165	0.1140	0.1140	0.1140	0.1136	0.1155	0.1140	0.1140	0.1140	0.1139
σ_z	0.1425	0.1430	0.1418	0.1425	0.1424	0.1350	0.1451	0.1425	0.1425	0.1425	0.1391	0.1443	0.1425	0.1425	0.1425	0.1408
z^n/σ_z		1.9313	2.2856	2.8570	2.4856	1.3998	2.6367	1.4091	4.5127	3.9261	1.7046	3.2497	5.0912	6.3639	5.5366	1.9396
Hubbard, Skinner and Zeldes' process, ρ=0.95, σ_ε^2=0.030																
ρ	0.9500	0.9577	0.9073	0.9998	0.9524	0.9563	0.9503	0.9394	0.9945	0.9496	0.9559	0.9499	0.9477	0.9759	0.9500	0.9532
σ_ε	0.1732	0.1843	0.1576	0.0101	0.1410	0.2221	0.1982	0.1670	0.0561	0.1692	0.1989	0.1883	0.1712	0.1198	0.1730	0.1874
σ_z	0.5547	0.6037	0.3275	0.5622	0.4792	0.5253	0.6205	0.4303	0.5556	0.5407	0.5415	0.5995	0.5043	0.5536	0.5536	0.5480
z^n/σ_z		1.9313	0.8921	2.8570	1.4079	1.3998	2.6367	1.4091	4.5127	2.2238	1.7046	3.2497	1.9871	6.3639	3.1361	1.9396
Storesletten, Telmer and Yaron's process, ρ=0.98, σ_ε^2=0.020																
ρ	0.9800	0.9952	0.9261	1.0000	0.9895	0.9989	0.9861	0.9619	1.0000	0.9815	0.9837	0.9810	0.9733	0.9998	0.9800	0.9823
σ_ε	0.1414	0.0838	0.1258	0.0000	0.0679	1.5788	0.1466	0.1332	0.0014	0.1177	0.1873	0.1634	0.1371	0.0142	0.1381	0.1672
σ_z	0.7107	0.7938	0.2782	0.7261	0.5468	0.9471	0.8448	0.3868	0.7173	0.6657	0.6938	0.8306	0.4924	0.7131	0.7041	0.7020
z^n/σ_z		1.9313	0.5685	2.8570	1.1521	1.3998	2.6367	0.8980	4.5127	1.8198	1.7046	3.2497	1.2664	6.3639	2.5663	1.9396

Note: 'Tauch.' is Tauchen's (1986) method and 'A–C' is Adda and Cooper's (2003) method. The first Tauchen–Hussey column uses the conditional variance and hence sets $\hat{\sigma}=\sigma_\varepsilon$, the second column uses the unconditional variance and sets $\hat{\sigma}=\sigma_z$ and the final column uses a weighted average of σ_ε and σ_z as described in the text.