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On the existence of symmetric mixed strategy equilibria

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Abstract

In this paper we provide a result ensuring the existence of symmetric mixed strategy equilibria in symmetric games. We apply the fixed point theorem of Glicksberg and Fan analogously to the way in which Moulin [Moulin, H., 1986. Game Theory for the Social Sciences, 2nd Edition. New York University Press, New York.] uses Kakutani's theorem to prove the existence of a symmetric equilibrium in pure strategies.

Keywords: Symmetric equilibria; Mixed strategies; Existence

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1. Introduction

The famous fixed point theorem of Glicksberg (1952) and Fan (1952) is often used in economic applications to guarantee the existence of mixed strategy Nash equilibria in games in

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which the (pure)strategy spaces are compact and convex, and the payoff functions are continuous.[1]

In this note we show that symmetric games satisfying these premises always possess a symmetric equilibrium in mixed strategies. We utilize Glicksberg (1952) fixed point theorem analogously to the way in which Moulin (1986, pp. 115–116) uses Kakutani's (1941) theorem to prove the existence of a symmetric equilibrium in pure strategies.

The result is useful for the analysis of games in which symmetric equilibria cannot be derived explicitly, and only some of their properties can be established.[2]

2. Setting and result

We consider a symmetric n-person game: Each player has the same (pure) strategy space A, which is assumed to be compact and Hausdorff.[3] Let the game be given in normal form by the continuous payoff function

$$R: \; A^n \rightarrow \mathbb{R},$$

where $R(x,y_1,\ldots,y_{n-1})$ is the payoff of an agent playing x, when the other $n-1$ agents play $y_1,\ldots,y_{n-1}$, respectively.[4] By S we denote the set of all mixed strategies over A, i.e. the set of all regular probability measures on A. From now on, S will be equipped with the weak topology. This implies that S is compact, since that is the case for A (see, for example, Bauer, 1992, p. 237).

The game's mixed strategy extension is described by the expected payoff function $\Re: \; S^n \rightarrow \mathbb{R}$, where

$$\Re(\sigma, \tau_1, \ldots, \tau_{n-1}) = \int \ldots \int \int R(x, y_1, \ldots, y_{n-1}) \mathrm{d}\sigma(x) \mathrm{d}\tau_1(y_1) \ldots \mathrm{d}\tau_{n-1}(y_{n-1})$$

is the expected payoff of an agent playing the mixed strategy $\sigma \in S$, the others playing $\tau_1,\ldots,\tau_{n-1}$, respectively.

The next theorem states our result.

Theorem 1 (*Existence*). *A symmetric game, as defined above, has a symmetric Nash equilibrium in mixed strategies*.

1 Mixed strategy equilibria are primarily discussed when the existence of equilibria in pure strategies cannot be guaranteed, mostly due to non-convexities in the best response correspondences. The examples are numerous and include auctions, tournaments, political contests, rent seeking, research and development races, models of imperfect competition, to mention but a few of them. Mixed strategy equilibria of, for example, the all-pay auction have been studied extensively, as these models do usually not possess pure strategy equilibria, but have many applications (see Baye et al., 1996; Hillman and Riley, 1989).

2 Burguet and Sakovics (1999), for example, study competition among auctioneers in setting the reserve price in second price auctions. As a symmetric mixed strategy equilibrium cannot be derived formally, they provide a discrete approximation. Damianov and Becker (2005) compare the supports of the symmetric mixed strategy equilibria of the uniform price and the discriminatory auction in a variable supply multi-unit auction model. The present theorem guarantees the existence of symmetric equilibria in these models.

3 Here we take the same general framework as Glicksberg; an applied economist will probably think of A as a compact subset of the Euclidean space $\mathbb{R}^m$. The theorem, however, applies to much more general cases, such as when strategies are whole supply or demand functions.

4 As the game is symmetric, $R(x,y_1,\ldots,y_{n-1})$ remains unchanged for all permutations of $y_1,\ldots,y_{n-1}$.