Finance Research Letters 000 (2015) 1–13

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

Longevity bond pricing under the threshold CIR model

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A R T I C L E I N F O

Article history:
Received 20 July 2015
Accepted 14 September 2015
Available online xxx

JEL classification:
C03
G12
G22

Keywords:
Mean reversion
Threshold CIR model
Longevity bonds
Longevity models
Interest rate

A B S T R A C T

While mean reversion is a well-documented feature in interest rate and commodity prices, empirical studies show that the long-term mean level and the mean reversion rate are not persistent in time. This paper introduces a threshold Cox–Ingersol–Ross (TCIR) model in which a regime shift is determined endogenously by the underlying financial asset. We derive the joint moment-generating function (MGF) of the terminal TCIR value and an average of it. The MGF enables us to value risk-free bonds and Longevity bonds analytically.

1. Introduction

Many financial variables, such as interest rates, volatility, mortality rates and commodity prices, exhibit both mean reversion and regime-switching feature. Although a financial variable will converge to its long-term equilibrium in the long run, the parameters involved usually take different values according to the economic regime.

Cox et al. (1985) (CIR) introduce a mean-reverting square-root process to model the stochastic interest rate market. The CIR model is particularly useful in the low interest rate environment as it ensures a positive value. Heston (1993) uses the CIR model to describe the stochastic volatility (SV). Wong and Lo (2009) investigate the valuation of options when the logarithm of the underlying asset is mean

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http://dx.doi.org/10.1016/j.frl.2015.09.010
1544-6123/

reverting with the Heston SV. Chiu et al. (2011) investigate the SV asymptotic of option pricing with mean reversion. The CIR model is often used in mortality modeling for pricing longevity bonds such as Blake et al. (2006, 2014); Bauer et al. (2010). Wong et al. (2014, 2015) investigate the hedging problem of longevity risk with cointegration, in which a special case corresponds to the CIR model. Wong and Zhao (2011b) show that the computation of interest rate derivatives involving a mean-reverting interest rate model could be numerically challenging.

However, the CIR model is considered practically inadequate for its constant parameter assumption. Hypothesis tests conducted by Bansal and Zhou (2002) sharply reject the CIR model with empirical data. Specifically, empirical evidence shows that parameters can change in different regimes. Two commonly used regime-switching models are the Markov-modulated and self-exciting threshold autoregressive models (Tong, 1983). The former assumes that the regime shift is driven by a hidden Markov chain while the latter assumes that the regime is determined by the region the underlying stochastic variable falls. Therefore, the regime is determined by an exogenous random variable in the Markov-modulated regime switching model while the threshold model determines regime shift endogenously through the asset value itself.

The regime-switching model is originated by Hamilton (1989) but there have been many extensions. Dai et al. (2007) develop and empirically implement an arbitrage-free, dynamic term structure model with "priced" factor and regime-shift risks based on Markov process. Wong and Zhao (2011a) assume the surplus of a firm follows an Ornstein–Uhlenbeck (OU) process with a surplus-dependent credit/debit interest rate, making the surplus process resembles a threshold OU process. Chi et al. (2015) investigate the valuation of European and barrier options under a threshold OU process. Milidonis et al. (2011) show that regime-switching is relevant to mortality modeling.

Generalizing Cox et al. (1985), we propose the threshold CIR (TCIR) model for positive financial variables by incorporating regime shift and mean reversion. The model is tractable in the sense that we manage to derive the joint moment-generating function (MGF) of the underlying asset and its average, $(X_T, \frac{1}{T}\int_0^T X_s ds)$. We show that the analytical expression of the MGF enables us to value zero-coupon bonds, longevity bonds and Asian options via numerical Laplace inversion.

Among the aforementioned financial products, we are particularly interested in the longevity bond, which is the building block of longevity securities. Tan et al. (2015) report that there is a reasonably large trading amount of longevity derivatives such as the longevity swaps in recent years. In particular, the longevity swap can be valued as a ratio of sums of longevity bonds.

The actuarial science literature uses to model the mortality rate by a positive mean reverting process and the CIR model is a popular approach. However, human mortality process seems to change according to medical innovation, economic growth and/or the occurrence of war. The TCIR model is a natural candidate for the mortality rate.

The remainder of the paper is organized as follows. Section 2 introduces the TCIR model, its statistical estimation, and offers empirical evidence from the data of interest rates, mortality rates and commodity prices. Section 3 derives the analytical solution to the joint MGF under 2-regime model and applies it to deduce analytical formulas for zero-coupon bond and longevity bond in terms of the Laplace transform. Section 4 present numerical studies and Section 5 concludes. Possible extension to n-regime model is presented in the appendix online.

2. The model

A non-negative stochastic process $\{X_t\}_{t\geq 0}$, such as interest rate, mortality rate and commodity price, is called the n-regime TCIR model if it is the solution of the following stochastic differential equation (SDE)

$$dX_t = \sum_{i=1}^{n}\{\kappa_i(\theta_i - X_t)dt + \sigma_i\sqrt{X_t}dW_t\}\mathbb{1}_{\{X_t\in R_i\}},\ X_0 = x, \tag{1}$$

where W_t is the Wiener process, $\mathbb{1}_A$ is the indicator function for the event A, and the parameters are defined as follows.

1. A sequence of threshold values: $\{0 = h_0 < h_1 < \cdots < h_n = \infty\}$ for $n \in \mathbb{N}$;