Finance Research Letters 000 (2016) 1–9

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

Optimal rates from eigenvalues

Peter Carr, Pratik Worah*

Courant Institute of Mathematical Sciences, 251 Mercer St, New York, NY 10016, United States

ARTICLE INFO

Article history:
Received 14 September 2015
Accepted 8 December 2015
Available online xxx

JEL Classification:
C5
C60
G10
G12

Keywords:
Interest rates
Partial differential equations

ABSTRACT

A financial portfolio typically pays dividend based on its value. We show that there is a unique portfolio that pays the maximum dividend rate while remaining solvent, under appropriate assumptions. We also give a characterization of both the portfolio and the optimal dividend rate.

1. Introduction

Suppose that one has a model for certain market variables and wants to construct a positive valued financial portfolio that is a function of those market variables and also pays a dividend at a constant yield of δ. It is natural to expect that if δ is too large than such a portfolio will go bankrupt (for us this simply means that its value would not remain positive). On the other hand, at the time of inception i.e. time zero, if one wants to create a market for such a portfolio[1] then one has to maximize that constant dividend yield δ. Assuming that the portfolio at hand depends on some finite tuple of "recurrent" market variables, like interest rates, inflation rates, ratio of commodity prices etc, we characterize the maximum dividend yield and the corresponding unique portfolio that provides this constant maximum dividend yield, without going bankrupt.

* Corresponding author. Tel.: +1 2177225641.
E-mail addresses: pcarr@nyc.rr.com (P. Carr), pworah@uchicago.edu (P. Worah).

[1] The example of ETFs come to mind, except that one has to model several other costs etc. to be realistic, we do not do that here.

http://dx.doi.org/10.1016/j.frl.2015.12.003
1544-6123/

More formally, consider a scenario where the value of a portfolio is simply a function of some economic uncertainty factor X_t. Suppose that the portfolio pays a continuous dividend at the constant rate of δ, which is to be determined at the formation of a portfolio. We show that if X_t is a strongly recurrent process (Donsker and Varadhan, 1976), for example a d-dimensional OU process, then the maximum dividend yield and the portfolio itself can be characterized by the eigenfunction of a certain second order PDE (see Theorem 2.1) .

As far as techniques in the paper are concerned, recently, Hansen and Scheinkman (2009) and Ross (2013) and others (for example Carr and Yu, 2012; Qin and Linetsky, 2014) have shown applications of principal eigenvalues and eigenfunctions and its appropriate generalizations to recovering market beliefs from option prices. Inspired by their ideas, in this note, our methods demonstrate an application of *generalized* principal eigenvalues (Pinsky, 1995) to our problem of calculating maximal dividend yields for financial portfolios.

We remark that Radner and Shepp (1996) and Jeanblanc-Piqué and Shiryaev (1995) did initiate study of the problem of determining optimal dividends for a given financial firm. However, their underlying model is different – theirs is an optimal control problem where the answer is again a dividend payment but it is a diffusion process that has to be optimized, while staving of insolvency indefinitely. Unlike our scenario, it is not a prespecified constant maximum dividend yield that has to be found while avoiding insolvency in the underlying model.

We also remark that, as a first guess, the reader may be tempted to think that the maximum constant dividend yield in question would always equal the long run average of the risk-free rate in the underlying model. However, this is only true if the risk-free rate is modeled as a constant stochastic process. Otherwise, large deviations in that stochastic process will lead to a different value for the maximum dividend yield.[2]

2. Our results

Our main result is summarized by the following statement:

Theorem 2.1. *Let X_t be an Itô process in $\mathbb{R}^d$ and assume that another Itô process $P(X_t)$ denotes the price, where $P(\cdot)$ is assumed positive and twice differentiable, of a financial portfolio P driven by X_t alone. Suppose P pays a continuously compounded dividend at the constant rate of δ. If X_t is a strongly recurrent process (see Definition 3.4) in the risk neutral measure then the maximum value of δ, for which such a P exists, is given by the critical eigenvalue of a certain elliptic PDE (see (Eq. 4.3)). Moreover, such an optimal portfolio is unique, as far as its dependence on X_t, and it corresponds to that positive eigenfunction of the same PDE.*

The crux of the proof of Theorem 2.1 relies on investigating when the PDE in Eq. (4.3) has a unique positive solution. The proof proceeds in two parts. First, we need to show the existence of a portfolio for the optimal dividend rate. The proof of this is based on elementary arguments and is given in Chapter 4 of the textbook by Pinsky (1995). The fundamental existence theorem here states that positive solutions exist at or above the critical eigenvalue but not below this critical eigenvalue (Theorem 4.1). This critical eigenvalue corresponds to the optimal dividend rate.[3] Second, we need to show that the portfolio for the optimal dividend rate is unique. The proof of this relies on the papers by Donsker and Varadhan (1976) on large deviation theory. The idea is to assume that there is more than one positive solution at the critical eigenvalue, these solutions are all candidates for the optimal portfolio, even when the underlying process is strongly recurrent. Next, we prove that the critical eigenvalue increases with the zeroth order term, when the underlying process is strongly recurrent (Lemma 4.5). Furthermore, when there is more than one positive solution at the critical eigenvalue, we construct a positive solution by perturbing the zeroth order term by a positive function, but for the same eigenvalue, which is now less than the critical eigenvalue of the perturbed PDE (Lemma 4.6).

[2] The reader may eventually check that the quantum harmonic oscillator can serve as a simple toy example to illustrate this remark.

[3] It will be the negation of the optimal dividend rate.