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Pure higher-order effects in the portfolio choice model

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ABSTRACT

This paper examines the effects of higher-order risk attitudes and statistical moments on the optimal allocation of risky assets within the standard portfolio choice model. We derive the expressions for the optimal proportion of wealth invested in the risky asset to show they are functions of portfolio returns third- and fourth-order moments as well as on the investor's risk preferences of prudence and temperance. We illustrate the relative importance that the introduction of those higher-order effects have in the decision of expected utility maximizers using data for the US.

1. Introduction

Following the theoretical contribution of Eeckhoudt and Schlesinger (2006) which set out lottery preference definitions, experimental studies were reported which examined the apportion of risks consistent with the higher-order risk preferences of prudence and temperance.[1] The reported experimental results revealed that a significant proportion of individuals make prudent and temperate choices consistent with standard expected utility theory (Ebert and Wiesen, 2014; Deck and Schlesinger, 2010).[2] Ebert and Wiesen (2014) provide an excellent review of the theoretical literature which examines the role of higher-order risk attitudes such as prudence and temperance on decision making in areas such as precautionary savings, monetary policy, insurance demand, and bidding in auctions. Despite these contributions, the solution of the classical portfolio choice model, i.e., the optimal proportion of wealth that an agent invests in the risky asset, has typically been obtained through a first-order Taylor approximation around a portfolio risk of zero (see Gollier (2001)). As a consequence the higher-order risk preferences play no role in portfolio choice that depends only on the mean and variance of returns and the investor's first- and second-order risk attitudes. To the best of our knowledge only the papers by

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[1] The importance of the third derivative of utility u ($u''' > 0$) in determining demand for precautionary savings defines prudence according to Kimball (1990). Behavioural aspects of investors have been related to the fourth-order derivative of the utility function ($u^{iv} < 0$) through the concept of temperance introduced by Kimball et al. (1992).

[2] Those experimental results are not surprising given that most commonly used expected utility theory functions imply prudent and temperate choices. These utility functions exhibit mixed risk aversion, i.e., nth-degree risk aversion for all orders (Ebert, 2013).

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Athayde and Flôres (2004) and Zakamouline and Koekebakker (2009) provide a closed-form solution up to the third-order moment for the portfolio choice model.[3] Otherwise models of optimal portfolio weights that incorporate higher-order effects have generally been obtained either as implicit solution (see Guidolin and Timmerman (2008); Jondeau and Rockinger (2006)) or using numerical optimization (see Kim et al. (2014)).

This article contributes to the literature by providing expressions for the optimal asset allocation in the classical portfolio problem that give an explicit role to the effects of higher-order investor's risk preferences of prudence and temperance as well as higher-order moments. We present an example employing US data to provide an intuition on the relative importance that the introduction of those higher-order effects could have to interpret investors decisions.

The remainder of the paper is organized as follows. Section 2 presents the standard portfolio choice model and our derivation of the optimal portfolio allocation using higher-order Taylor approximations. Section 3 is an illustrative example of the model using actual data for the US. Section 4 summarizes the conclusions.

2. Higher-order risk preferences in the classical portfolio choice model

Consider an investor with a utility function u and initial wealth W that she can invest in risk-free and risky assets. Let r and $\widetilde{x}_0$ be the after-one-period sure and random net return of risk-free and risky assets, respectively. The problem of the agent is to choose the optimal amount of initial wealth invested in the risky asset, α, that maximizes her expected utility $V(\alpha)$

$$\underset{\{\alpha\}}{Max}\, V(\alpha) = Eu(\omega_0 + \alpha\widetilde{x}), \tag{1}$$

where $\widetilde{x} = \widetilde{x}_0 - r$ is the excess return, $\omega_0 = W(1+r)$ and $\alpha\widetilde{x}$ are after-one-period sure and random wealth, respectively. To determine the solution of Eq. (1) we assume that the portfolio risk, k, is small, and as k is endogenous, we define the excess return, as is standard, as $\widetilde{x} = k\mu + \widetilde{y}$, where $E\widetilde{y} = 0$, $\mu > 0$, and $E\widetilde{x}$ is the risk premium.[4]

In order to employ the relevant information contained in returns' moments and investor's risk preferences up to the fourth-order, we use a 3rd-order Taylor expansion of $\alpha^*(k)$ around $k = 0$, after some calculations we obtain the optimal portfolio weight as:[5,6]

$$\begin{aligned}\alpha^*_{(3)} \simeq & \left(\frac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5} \frac{P(\omega_0)^2}{A(\omega_0)^3} + \left(\frac{4}{3(V\widetilde{x})^2} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3} \right) \frac{P(\omega_0)}{A(\omega_0)^2} - \frac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4} \frac{T(\omega_0)P(\omega_0)}{A(\omega_0)^3} - \frac{1}{A(\omega_0)(V\widetilde{x})^2} \right) (E\widetilde{x})^3 \\ & + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P(\omega_0)}{A(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{A(\omega_0)V\widetilde{x}} E\widetilde{x} \\ & = Z(\cdot)(E\widetilde{x})^3 + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P(\omega_0)}{A(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{A(\omega_0)V\widetilde{x}} E\widetilde{x}, \end{aligned} \tag{2}$$

where $A(\omega_0) = -Eu''(\omega_0)/Eu'(\omega_0)$, is the Arrow–Pratt index of absolute risk aversion, $P(\omega_0) = -Eu'''(\omega_0)/Eu''(\omega_0)$ and $T(\omega_0) = -Eu^{iv}(\omega_0)/Eu'''(\omega_0)$ are the investor's degree of absolute prudence and temperance, respectively, $V\widetilde{x}$ denotes the variance of $\widetilde{x}$, and $E(\widetilde{x} - E\widetilde{x})^3$ and $E(\widetilde{x} - E\widetilde{x})^4$ are the third- and fourth-order central moments of $\widetilde{x}$, respectively. We note that $Z(\cdot)$ is a function of $\widetilde{x}$'s four first-order moments and investor's risk preferences up to temperance.[7]

By dividing Eq. (2) by sure wealth, ω_0, we obtain the 3rd-order Taylor approximated optimal share of the portfolio invested in the risky asset as

$$\begin{aligned}\alpha^*_{P,(3)} \simeq & \left(\frac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5} \frac{P_R(\omega_0)^2}{R(\omega_0)^3} + \left(\frac{4}{3(V\widetilde{x})^2} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3} \right) \frac{P_R(\omega_0)}{R(\omega_0)^2} - \frac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4} \frac{T_R(\omega_0)P_R(\omega_0)}{R(\omega_0)^3} - \frac{1}{R(\omega_0)(V\widetilde{x})^2} \right) (E\widetilde{x})^3 \\ & + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P_R(\omega_0)}{R(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{R(\omega_0)V\widetilde{x}} E\widetilde{x} \\ & = Z_R(\cdot)(E\widetilde{x})^3 + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P_R(\omega_0)}{R(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{R(\omega_0)V\widetilde{x}} E\widetilde{x}, \end{aligned} \tag{3}$$

where $Z_R(\cdot)$ depends on distributional moments of $\widetilde{x}$ up to the fourth order, and the relative measures of risk aversion, $R(\omega_0) = \omega_0 A(\omega_0)$, prudence, $P_R(\omega_0) = \omega_0 P(\omega_0)$, and temperance, $T_R(\omega_0) = \omega_0 T(\omega_0)$.

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[3] This approach allows Zakamouline and Koekebakker to present a theoretically sound portfolio performance measure that takes into account the skewness of the distribution of returns. Their Adjusted for Skewness Sharpe Ratio has a direct relation to the level of expected utility provided by the asset. This is in contrast to many other *arbitrary* –not theoretically founded– reward-to-risk ratios such as performance measures based on Value-at-Risk. In Athayde and Flôres (2004) the Markowitz's efficient frontier is extended to a three-moments multidimensional portfolio choice framework.

[4] k may be negative, i.e., the model allows a short-sale of the risky asset; see proposition 6 in Gollier (2001, p. 54).

[5] Note that the optimal investment in the risky asset, $\alpha^*(k)$, depends on k, so if $E\widetilde{x} = 0$, i.e., $k = 0$, it is optimal to invest 0 units of wealth in the risky asset; $\alpha^*(0) = 0$. $\alpha^*(k)$ is obtained assuming $k > 0$.

[6] As is usual we assume that the moments of $\widetilde{y}$ are constant, i.e., $E\widetilde{y}^n u^{(n)}(\omega_0) = E\widetilde{y}^n Eu^{(n)}(\omega_0)\ \forall n$.

[7] To save space, the derivation of expression (2) is provided in the Appendix.