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# The intrinsic bounds on the risk premium of Markovian pricing kernels ☆

Jihun Han, Hyungbin Park *

*Courant Institute of Mathematical Sciences, New York University, New York, NY, USA*

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ABSTRACT

The risk premium is one of main concepts in mathematical finance. It is a measure of the trade-offs investors make between return and risk and is defined by the excess return relative to the risk-free interest rate that is earned from an asset per one unit of risk. The purpose of this article is to determine upper and lower bounds on the risk premium of an asset based on the market prices of options. One of the key assumptions to achieve this goal is that the market is Markovian. Under this assumption, we can transform the problem of finding the bounds into a second-order differential equation. We then obtain upper and lower bounds on the risk premium by analyzing the differential equation.



## 1. Introduction

The *risk premium* or *market price of risk* is one of main concepts in mathematical finance. The risk premium is a measure of the trade-offs investors make between return and risk and is defined by the excess return relative to the risk-free interest rate earned from an asset per one unit of risk. The risk premium determines the relation between an objective measure and a risk-neutral measure. An objective measure describes the actual stochastic dynamics of markets, and a risk-neutral measure determines the prices of options.

☆ This document is a collaborative effort.

* Corresponding author.
*E-mail addresses:* jihunhan@cims.nyu.edu (J. Han), hyungbin@cims.nyu.edu, hyungbin2015@gmail.com (H. Park).

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Recently, many authors have suggested that the risk premium (or, equivalently, objective measure) can be determined from a risk-neutral measure. Ross (2013) demonstrated that the risk premium can be uniquely determined by a risk-neutral measure. His model assumes that there is a finite-state Markov process $X_t$ that drives the economy in discrete time $t \in \mathbb{N}$. Many authors have extended his model to a continuous-time setting using a Markov diffusion process $X_t$ with state space $\mathbb{R}$; see, e.g., Borovicka et al. (2011, 2014), Carr and Yu (2012), Dubynskiy and Goldstein (2013), Park (2014b), Qin and Linetsky (2014b) and Walden (2013). Unfortunately, in the continuous-time model, the risk premium is not uniquely determined from a risk-neutral measure; Goodman and Park (2014), Park (2014b).

To determine the risk premium uniquely, all of the aforementioned authors assumed that some information about the objective measure was known or restricted the process $X_t$ to some class. Borovicka et al. (2014) made the assumption that the process $X_t$ is *stochastically stable* under the objective measure. Carr and Yu (2012) assumed that the process $X_t$ is a *bounded* process. Dubynskiy and Goldstein (2013) explored Markov diffusion models with *reflecting boundary* conditions. Park (2014b) assumed that $X_t$ is non-attracted to the left (or right) boundary under the objective measure. Qin and Linetsky (2014b) and Walden (2013) assumed that the process $X_t$ is *recurrent* under the objective measure. Without these assumptions, one cannot determine the risk premium uniquely. As a closely related topic, refer to Davydov and Linetsky (2003), Gorovoi and Linetsky (2004), Linetsky (2004), Park (2014a) and Qin and Linetsky (2014a).

The purpose of this article is to investigate the bounds of the risk premium. As mentioned above, without further assumptions, the risk premium is not uniquely determined, but one can determine upper and lower bounds on the risk premium. To determine these bounds, we need to consider how the risk premium of an asset is determined in a financial market.

A key assumption of this article is that the reciprocal of the pricing kernel is expressed in the form $e^{\beta t}\,\phi(X_t)$ for some positive constant $\beta$ and positive function $\phi(\cdot)$. For example, in the *consumption-based capital asset model* in Bansal and Yaron (2004), Breeden (1979), Campbell and Cochrane (1999) and Karatzas and Shreve (1998), the pricing kernel is expressed in the above form. We will see that in this case the risk premium $\theta_t$ is given by

$$\theta_t = (\sigma\phi'\phi^{-1})(X_t), \tag{1.1}$$

where $\sigma(X_t)$ is the volatility of $X_t$.

The problem of determining the bounds of the risk premium can be transformed into a second-order differential equation. We will demonstrate that $\phi(\cdot)$ satisfies the following differential equation:

$$\mathcal{L}\phi(x) := \frac{1}{2}\sigma^2(x)\phi''(x) + k(x)\phi'(x) - r(x)\phi(x) = -\beta\;\phi(x)$$

for some unknown positive number $\beta$. Thus, we can determine the bounds of the risk premium by investigating the bounds of $(\sigma\phi'\phi^{-1})(\cdot)$ for a positive solution $\phi(\cdot)$. It will be demonstrated that two special solutions of $\mathcal{L}h = 0$ play an important role for the bounds of the risk premium $\theta_t$.

The following provides an overview of this article. In Section 2, we state the notion of Markovian pricing kernels. In Section 3, we investigate the risk premium of an asset and see how the problem of determining the bounds of the risk premium is transformed into a second-order differential equation. In Section 4, we find upper and lower bounds on the risk premium of an asset, which is the main result of this article. In Section 5, we see how this result can be applied to determine the range of return of an asset. Finally, Section 6 summarizes this article.

## 2. Markovian pricing kernels

A financial market is defined as a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having a Brownian motion $B_t$ with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ generated by $B_t$. All the processes in this article are assumed to be adapted to the filtration $\mathcal{F}$. $\mathbb{P}$ is the objective measure of this market.

**Assumption 1.** In the financial market, there are two assets. One is a *money market account* $e^{\int_0^t r_s\,ds}$ with an *interest rate* process $r_t$ and the other is a risky asset $S_t$ satisfying $dS_t = \mu_t S_t\,dt + v_t S_t\,dB_t$.

Throughout this article, the stochastic discount factor is the money market account.