Finance Research Letters 13 (2015) 54–65

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

# Modelling default risk with occupation times

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ARTICLE INFO

*Article history:*
Received 20 February 2015
Accepted 23 March 2015
Available online 3 April 2015

*JEL classification:*
G12
G13
G19
G33

*Keywords:*
Occupation time
Credit risk
Merton model
Black–Cox model
Structural models

ABSTRACT

This paper develops a semi-analytic pricing formula, easily implemented via quadrature, for a structural model based on occupation times that contains both the Merton and Black–Cox models as limiting cases. In the model liquidation is triggered as soon as the firm's asset value has spent a prespecified amount of time below the default barrier. Surprisingly, we find that the value of the firm's debt (i) need not be monotone in the length of the grace period and (ii) need not lie between the limiting Merton and Black–Cox values.


## 1. Introduction

Structural models based on occupation times or excursion times allow for a temporal separation between the default event and liquidation. As such, they allow for a more faithful representation of actual bankruptcy proceedings. See, for instance, the discussions in Moraux (2002), Nardon (2008), Galai et al. (2007), Yildirim (2006), Li et al. (2014) and Broadie et al. (2007). Unfortunately, when occupation or excursion times are introduced there is often little hope of analytic pricing formulae (see Broadie et al. (2007) for a general-purpose lattice method). For instance in the context of occupation times, Nardon (2008) only considers computation of default probabilities and does not address bond valuation while the analytic formula provided in Moraux (2002) involves an unrealistic terminal payoff for the firm's debt.

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http://dx.doi.org/10.1016/j.frl.2015.03.003
1544-6123/

This paper develops a semi-analytic pricing formula, easily implemented via quadrature, for a structural model based on occupation times that contains both the Merton (1974) and Black and Cox (1976) models as limiting cases. We introduce a process that keeps track of the cumulative amount of time that the firm's asset value spends below the default barrier, and assume that liquidation is triggered when this occupation time first reaches a predefined upper threshold. In keeping with the aforementioned literature we may interpret this parameter as the length of a grace period granted by the bankruptcy court, during which reorganization is possible and beyond which liquidation is enforced. The Merton and Black–Cox models are obtained by sending this threshold parameter to its upper and lower limit, respectively. Surprisingly, we find that the value of the firm's debt (i) need not be monotone in the threshold parameter and (ii) need not lie between the Merton and Black–Cox values.

## 2. Model

We consider a firm whose capital structure consists of common equity and zero-coupon debt, and whose asset value evolves as geometric Brownian motion. Thus if $V_t$ denotes the total value of the firm's assets at time $t$ then $V_t = V_0 \exp([r - q - \sigma^2/2]t + \sigma W_t)$, where the risk-free rate of interest $r$, the dividend yield $q$ and the volatility $\sigma$ are constant, and $\{W_t\}_{t \geqslant 0}$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}}, \mathbb{F})$. We assume that the firm's outstanding debt consist of a single zero-coupon bond with a face value of $F$ dollars and maturity date $T$.

As in the Black–Cox model we introduce a default barrier $L(t) := K\mathrm{e}^{-\gamma(T-t)}$, where $K < \min(V_0 \mathrm{e}^{\gamma T}, F)$ is a constant, and say that the firm is in default at time $t$ if $V_t < L(t)$. In a traditional first-passage model the firm would be liquidated at time $\tau$, provided $\tau < T$, where

$$\tau := \inf\{t \geqslant 0 : V_t \leqslant L(t)\}. \tag{1}$$

In the model proposed here we do not assume that default immediately triggers liquidation. Instead we assume that the firm is allowed to continue operating even if it is in default, provided it has not spent too much time in the default state. Formally, we introduce the (positive, non-decreasing) occupation time process

$$\mathcal{A}_t := \int_0^t \mathbb{1}_{\{V_u \leqslant L(u)\}} \mathrm{d}u \tag{2}$$

and assume that liquidation is triggered as soon as $\mathcal{A}_t \geqslant \vartheta$ for the first time, where $\vartheta \in [0, T]$ is a prespecified constant. In other words, if $\tau_\vartheta < T$ then the firm is liquidated at time $\tau_\vartheta$, where

$$\tau_\vartheta := \inf\{t \geqslant 0 : \mathcal{A}_t \geqslant \vartheta\}. \tag{3}$$

Conceptually we may imagine that, upon entering the default state for the first time, the firm is given a grace period in which to "right the ship", and we may interpret the threshold parameter $\vartheta$ as the length of the grace period. The stopping time $\tau_\vartheta$ and the occupation time $\mathcal{A}_t$ are related as follows:

$$\{\tau_\vartheta \geqslant t\} = \{\mathcal{A}_t \leqslant \vartheta\} \text{ for any } t \geqslant 0. \tag{4}$$

A sample path of the occupation time $\mathcal{A}_t$ is provided in Fig. 1, where it is also demonstrated how the first hitting time $\tau$ and the corresponding first occupation time $\tau_\vartheta$ are calculated for a sample path.

If the firm avoids being liquidated and is solvent at maturity then bondholders receive $F$ at maturity $T$. If the firm avoids being liquidated but is insolvent at maturity, bondholders receive $\beta_1 V_T$ at maturity, where $\beta_1 \in [0, 1]$ is a some constant that could potentially reflect bankruptcy costs. Finally, if the firm is liquidated prior to maturity then bondholders receive $\beta_2 V_{\tau_\vartheta}$ at the liquidation time $\tau_\vartheta$, where $\beta_2 \in [0, 1]$ is a constant. Thus, when viewed as a contingent claim on asset value, the bond is characterized the following payoff received at time $\tau_\vartheta \wedge T \equiv \min(\tau_\vartheta, T)$:

$$D_{\tau_\vartheta \wedge T} := F\mathbb{1}_{\{V_T \geqslant F, \tau_\vartheta \geqslant T\}} + \beta_1 V_T \mathbb{1}_{\{V_T < F, \tau_\vartheta \geqslant T\}} + \beta_2 V_{\tau_\vartheta} \mathbb{1}_{\{\tau_\vartheta < T\}} \tag{5}$$

$$= F\mathbb{1}_{\{V_T \geqslant F, \mathcal{A}_T \leqslant \vartheta\}} + \beta_1 V_T \mathbb{1}_{\{V_T < F, \mathcal{A}_T \leqslant \vartheta\}} + \beta_2 V_{\tau_\vartheta} \mathbb{1}_{\{\mathcal{A}_T > \vartheta\}}. \tag{6}$$