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Case study

Non-singular expressions for the spherical harmonic synthesis of gravitational curvatures in a local north-oriented reference frame

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ABSTRACT

Third-order gradients of the gravitational potential (gravitational curvatures) have already found some applications in geosciences. Observability of these parameters, describing the Earth's gravitational field in a more complex way than any other currently available gravitational parameter, such as gravitational acceleration (first-order gradient) or gravitational (second-order) gradient, is currently discussed by physicists. Moreover, first designs of observational devices (sensors) have already been proposed. The spherical harmonic analysis and synthesis are the common tools used by geoscientists to study spectral properties of various functionals of the Earth's gravitational potential. However, the conventional spherical harmonic expansions of the gravitational curvatures in the local north-oriented reference frame have rather complicated forms that depend on the first-, second- and third-order derivatives of the associated Legendre functions. Moreover, some of these expansions also contain singular terms at the poles. In this paper, the conventional series are transformed to new simpler and non-singular forms based on relations between the associated Legendre functions and their derivatives. Numerical experiments demonstrate the applicability and correctness of the new expressions.

1. Introduction

Currently available ground, marine, airborne and satellite sensors for Earth's gravity field mapping allow for gravity or gravitational acceleration measurements. The relative accuracy of ground gravity measurements reached the ppb level. However, the accuracy of measurements performed at moving platforms is significantly worse. Moreover, the full signal cannot usually be recovered due to extensive data filtering in order to reduce the observation noise. If the full gravitational acceleration vector is measured (vector gravimetry), we can recover all components of the first-order gravitational tensor that consists of three gravitational gradients in a specific coordinate frame. By combining gravitational acceleration measurements the second-order gravitational tensor can be derived. This is basically the observational principle of the gravity-dedicated satellite mission GOCE (Gravity field and steady-state Ocean Circulation Explorer), see, e.g., ESA (1999) and Rummel (2010). Gravitational observables and/or derived pseudo-observables have widely been exploited by geoscientists for the purpose of the Earth's gravitational field modelling, interpretation and spectral analyses.

In recent years, new sensors for observing a third-order gravitational tensor have been proposed. The Russian project Dulkyn (www.dulkyn.ru) aims at developing a system that would eventually observe third-order directional derivatives of the gravitational potential (in case of the Earth's gravitational field also called shortly geopotential) together with their temporal variations (Balakin et al., 1997). Recently, Rosi et al. (2015) performed first measurements of the third-order vertical derivatives of the geopotential at the Earth's surface. The gravity-dedicated satellite mission called OPTIMA (OPTical Interferometry for global MAss change detection from space) designed to measure the third-order derivatives of the geopotential was proposed by Brieden et al. (2010). Motivated by higher sensitivity of the third-order derivatives of the geopotential to short-wavelength structures of the Earth's gravitational field (Jacoby and Smilde, 2009), their exploitation has repeatedly been suggested for geophysical exploration purposes, see, e.g., Troshkov and Shalaev (1968), Smith et al. (1998), Fedi and Florio (2001), Thurston et al. (2002), Abdelrahman et al. (2003), Hafez et al. (2006), Pajot et al. (2008), Veryaskin and McRae (2008), Beiki (2010) and Eppelbaum (2011).

These ongoing efforts opened a new chapter in the area of observability of gravitation (gravity stripped of centrifugal acceleration). In geodesy, the third-order derivatives of the geopotential have been discussed in various contexts. Already Moritz (1967) investigated parameters of the Earth's gravitational field up to the third-order gravitational tensor and showed that gravitational tensors of the order three and higher were independent of sensors'

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orientation. Thus, any instrument capable of observing such quantities would provide a pure gravitational signal. Rummel (1986), Rummel et al. (1993), Koop (1993) and Albertella et al. (2000) used the third-order gravitational tensor for the error analysis of gradiometric observations. Ardalan and Grafarend (2001) expanded the normal gravitational potential (generated by a homogeneous geocentric biaxial ellipsoid) into the Taylor series up to the third-order geopotential derivatives in analyzing Bruns's formula. Grafarend (1997) also derived a functional relationship between curvature and torsion of a plumb-line to second- and third-order derivatives of the geopotential. Casotto and Fantino (2009) derived expressions for gravitational tensors up to the third order in local and global reference frames by tensor analysis. Most recently, Šprlák and Novák (2015) studied functional relationships between third-order geopotential derivatives and gravitating mass density distribution, anomalous gravitational acceleration and the geopotential.

Spectral properties of the gravitational field based on the third-order derivatives of the geopotential have also been studied. Cunningham (1970) derived spherical harmonic series for gravitational tensors of an arbitrary order in a geocentric reference frame. This study was extended by Metris et al. (1999) and Petrovskaya and Vershkov (2010). Computational aspects of the harmonic synthesis up to the third-order derivatives of Legendre's functions were discussed by Fantino and Casotto (2009) and Fukushima (2012, 2013). Non-singular expressions for a geomagnetic vector and gradient tensor fields were also studied by Du et al. (2015). Expressions for both the spherical harmonic analysis and synthesis use associated Legendre's functions of the first kind. Expressions for the third-order derivatives of the geopotential in spherical coordinates include respective derivatives of the associated Legendre functions. Moreover, they include terms dependent on latitude which are singular at both poles.

In this paper recursive expressions for computing values of the third-order derivatives of the associated Legendre functions are given. Formulas for spherical harmonic synthesis are modified to avoid numerical instabilities including singularities at the poles. A simple analytical structure of the new expressions is particularly suitable for deriving geopotential coefficients from eventually available observables as well as for studying spectral properties of the Earth's gravitational field based on the third-order gradients of the geopotential. Conventional expansions for the gravitational curvatures in a local north-oriented reference frame (LNOF) are transformed on the basis of relations given by Ilk (1983), Petrovskaya and Vershkov (2006) and Eshagh (2008, 2010).

The paper is organized as follows: in Section 2 we formulate the problem, define differential operators and provide conventional and new non-singular expressions for the series representation of the components of the third-order gravitational tensor in LNOF; in Section 3 we provide underlying expressions for derivatives of the associated Legendre functions and outline derivation of the new non-singular expressions; Section 4 contains numerical results obtained through computer realizations of the new formulas that demonstrate their correctness and functionality at the poles; finally, contributions of the paper are summarized in Conclusions. We also provide a MATLAB based program for potential users.

2. Formulation of the problem

In the following we define the Earth's disturbing potential T as a difference of the Earth's gravitational potential reduced for the gravitational potential of the geocentric homogeneous biaxial ellipsoid (such as GRS80, Moritz, 2000). Outside the Earth's masses (and the reference ellipsoid), the disturbing potential T is a harmonic function as it satisfies the Laplace–Poisson differential equation. Thus, it can be represented by a spherical harmonic series of the form, e.g., Heiskanen and Moritz (1967, Section 2–14):

$$T(r,\varphi,\lambda) = \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\bar{P}_{n,m}(\sin\varphi)\,(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda). \tag{1}$$

In this equation, GM represents the geocentric gravitational constant, a is the radius of a mean geocentric sphere approximating the Earth, $\bar{P}_{n,m}$ is the fully normalized associated Legendre function of the first kind of degree n and order m, and $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are respective fully normalized spherical harmonic coefficients. The disturbing potential T is a function of three coordinates (neglecting its temporal variability) that define its location in 3-D space: in our particular case they include geocentric radius r, spherical latitude φ and longitude λ. As available observations (and numerical limitations) allow for determination only of a finite number of the spherical harmonic coefficients, the series is truncated at some maximum degree N (currently at the level of $\approx$ 2000).

The third-order gravitational tensor is represented by 27 gravitational third-order geopotential gradients (gravitational curvatures) but only 10 of them are distinct from each other because of continuity of the Earth's gravitational field. In this paper, we will consider only gravitational curvatures referred to LNOF. Such a reference frame is defined by an origin in the point of interest and by a right-handed orthogonal basis with the following orientation of axes: the x-axis points to the North, the y-axis points to the West and the z-axis is directed radially outward. Each of these ten gravitational curvatures is defined by one differential operator. Such differential operators in terms of the spherical geocentric coordinates read as follows (Tóth, 2005; Casotto and Fantino, 2009):

$$\mathcal{D}^{xxx} = -\frac{1}{r^2}\left(\frac{2}{r}\frac{\partial}{\partial\varphi} - 3\frac{\partial^2}{\partial r\partial\varphi} - \frac{1}{r}\frac{\partial^3}{\partial\varphi^3}\right), \tag{2}$$

$$\mathcal{D}^{xxy} = -\frac{1}{r^2\cos\varphi}\left(\frac{2\tan^2\varphi}{r}\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial r\partial\lambda} + \frac{2\tan\varphi}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} + \frac{1}{r}\frac{\partial^3}{\partial\varphi^2\partial\lambda}\right), \tag{3}$$

$$\mathcal{D}^{xxz} = -\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial^3}{\partial r\partial\varphi^2}\right), \tag{4}$$

$$\mathcal{D}^{xyy} = -\frac{1}{r^2}\left(\frac{1}{r\cos^2\varphi}\frac{\partial}{\partial\varphi} - \frac{\partial^2}{\partial r\partial\varphi} + \frac{\tan\varphi}{r}\frac{\partial^2}{\partial\varphi^2} - \frac{2\tan\varphi}{r\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{r\cos^2\varphi}\frac{\partial^3}{\partial\varphi\partial\lambda^2}\right), \tag{5}$$

$$\mathcal{D}^{xyz} = \frac{1}{r^2\cos\varphi}\left(\frac{2\tan\varphi}{r}\frac{\partial}{\partial\lambda} - \tan\varphi\frac{\partial^2}{\partial r\partial\lambda} + \frac{2}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} - \frac{\partial^3}{\partial r\partial\varphi\partial\lambda}\right), \tag{6}$$

$$\mathcal{D}^{xzz} = \frac{1}{r}\left(\frac{2}{r^2}\frac{\partial}{\partial\varphi} - \frac{2}{r}\frac{\partial^2}{\partial r\partial\varphi} + \frac{\partial^3}{\partial r^2\partial\varphi}\right), \tag{7}$$

$$\mathcal{D}^{yyy} = \frac{1}{r^2\cos\varphi}\left(\frac{2}{r\cos^2\varphi}\frac{\partial}{\partial\lambda} - 3\frac{\partial^2}{\partial r\partial\lambda} + \frac{3\tan\varphi}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} - \frac{1}{r\cos^2\varphi}\frac{\partial^3}{\partial\lambda^3}\right), \tag{8}$$