Games and Economic Behavior ••• (••••) •••–•••

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

# Ordinal and cardinal solution concepts for two-sided matching ☆

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ARTICLE INFO

*Article history:*
Received 1 May 2014
Available online xxxx

*JEL classification:*
C72
C78

*Keywords:*
Market design
Matching theory
National resident matching program

ABSTRACT

We characterize solutions for two-sided matching, both in the transferable- and in the nontransferable-utility frameworks, using a cardinal formulation. Our approach makes the comparison of the matching models with and without transfers particularly transparent. We introduce the concept of a no-trade stable matching to study the role of transfers in matching. A no-trade stable matching is one in which the availability of transfers does not affect the outcome.



## 1. Introduction

We explore the role of transfers and cardinal utility in matching markets. Economists regularly use one- and two-sided models, with and without transfers. For example auctions allow for monetary transfers among the agents, while models of marriage, organ donation and "housing" exchanges do not. There are two-sided matching models of the labor market without transfers, such as the market for medical interns in the US; and traditional models of the labor market where salaries, and therefore transfers, are allowed. We seek to understand how and why transfers matter in markets for discrete goods.

The question is interesting to us as theorists, but it also matters greatly for one of the most important applications of matching markets, namely the medical interns market. In the market for medical interns in the US (see Roth, 1984a; Roth and Sotomayor, 1990; Roth, 2002), hospitals match with applicants using a centralized clearinghouse that implements a stable matching. We always think of this market as one *without* transfers, because salaries are fixed first, before the matching is established. So at the stage in which the parties "bargain" over who is to be matched to whom, salaries are already fixed, and thus there are no transfers.

☆ The authors wish to thank Juan Pereyra Barreiro and Aditya Kuvalekar for many useful suggestions, and for pointing out a mistake in one of the examples in a previous version of the paper.

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[2] This research has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013) / ERC grant agreement n° 313699.

http://dx.doi.org/10.1016/j.geb.2015.10.002
0899-8256/

Please cite this article in press as: Echenique, F., Galichon, A. Ordinal and cardinal solution concepts for two-sided matching. Games Econ. Behav. (2015), http://dx.doi.org/10.1016/j.geb.2015.10.002

There is a priori no reason for things to be this way. Hospitals and interns could instead bargain over salaries and employment at the same time. This is arguably the normal state of affairs in most other labor markets; and it has been specifically advocated for the medical interns market in the US (see Crawford, 2008). It is therefore important to understand the impact of disallowing transfers in a matching market. Our paper is a first step towards understanding this problem.

In a two-sided matching market – for our purposes, in the Gale–Shapley marriage market – this impact is important. We consider two canonical models: the marriage market without transfers (the NTU model) and the marriage market with transfers (the TU model, also called the assignment game).

There are Pareto efficient, and even stable, matchings in the NTU model that a utilitarian social planner would never choose, regardless of how she weights agents' utilities. Implicitly, a utilitarian social planner has access to transfers. Our results motivate an investigation into the distance between the utilitarian welfare in the presence of transfers, and the utilitarian welfare in the absence of transfers. We show that this gap can be arbitrarily large. In fact, it can grow exponentially with the size of the market. Our result should be contrasted with Lee and Yariv (2014), who show that the gap between utilitarian welfare and stability disappears in large markets. Lee and Yariv impose certain regularity conditions on preferences; or result is a "worst case" analysis, showing that there are markets for which the difference between efficiency and stability can be arbitrarily large. Of course, our result is established in an environment in which utilities are bounded, and the bound is constant while the market grows (otherwise the exercise would be meaningless). See our Proposition 16.

From the viewpoint of the recent literature in computer science on the "price of anarchy" (see e.g. Roughgarden, 2005), Proposition 16 says that the "price of no transfers" can be arbitrarily bad, and grow super-exponentially with the size of the market. In that sense, our result avoids the main critique that economists often levy on the price of anarchy literature. Many papers on the price of anarchy compare equilibria with the utilitarian efficient outcomes, in models in which there are no transfers allowed. Our paper deals with a model in which transfer play a role: namely the assignment game, or the TU model of matching. The meaning then of our price of transfer result is that the loss in utilitarian welfare from banning transfers (and thus moving from the assignment game to the NTU matching model) is large and grows superexponentially with the size of the market. See Section 4.2 for details.

We present results characterizing Pareto efficiency and the role of transfers in marriage models. Ex-ante Pareto optimality in the model with transfers is characterized by the maximization of the weighted utilitarian sum of utilities, while Pareto optimality when there are no transfers is equivalent to a different maximization problem, one where the weighted sum of "adjusted" utilities are employed. Each of these problems, in turn, have a formulation as a system of linear inequalities. The results follow (perhaps unexpectedly) from Afriat's theorem in the theory of revealed preference.

In order to explore the role of transfers, we study a special kind of stable matching: A *no-trade stable matching* in a marriage market is a matching that is not affected by the presence of transfers. This is the central notion in our paper. Agents are happy remaining matched as specified by the matching, even if transfers are available, and *even though they do not make use of transfers.* Transfers are available, but they are not needed to support the stable matching. There is thus a clear sense in which transfers play no role in a no-trade stable matching.

The notion of no-trade stable matching is useful for the following reason. We can think of transfers as making some agents better off at the expense of others. It is then possible to modify a market by choosing a cardinal utility representation of agents preferences with the property that the matching remains stable with and without transfers (Theorem 12). Under certain circumstances, namely when the stable matchings are "isolated," we can choose a cardinal representation that will work in this way for every stable matching. So under such a cardinal representation of preferences, any stable matching remains stable regardless of the presence of transfers. Finally (Example 14), we cannot replicate the role of transfers by re-weighting agents' utilities. In general, to instate a no-trade stable matching, we need the full freedom of choosing alternative cardinal representations.

It is easy to generate examples of stable matchings that cannot be sustained when transfers are allowed, and of stable matchings that can be sustained with transfers (in the sense of being utilitarian-efficient), but where transfers are actually used to sustain stability. We present conditions under which a market has a cardinal utility representation for which stable matchings are no trade matchings.

In sum, the notion of a no-trade stable matching captures both TU and NTU stability: a no-trade stable matching is also a TU and NTU stable matching. TU stability is, on the other hand, strictly stronger than ex-ante Pareto efficiency, which is strictly stronger than ex-post Pareto efficiency. NTU stability is strictly stronger than ex-post Pareto efficiency.[3]

The model without transfers was introduced by Gale and Shapley (1962). The model with transfers is due to Shapley and Shubik (1971). Kelso and Crawford (1982) extended the models further, and in some sense Kelso and Crawford's is the first paper to investigate the effects of adding transfers to the Gale–Shapley marriage model. Roth (1984b) and Hatfield and Milgrom (2005) extended the model to allow for more complicated contracts, not only transfers (see Hatfield and Kojima, 2010; Echenique, 2012 for a discussion of the added generality of contracts). We are apparently the first to consider the

[3] TU and NTU stability are not comparable in this sense. Empirically, though, they are comparable, with TU stability having strictly more testable implications than NTU stability (Echenique et al., 2013).

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