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Strategy-proof and fair assignment is wasteful ☆

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ABSTRACT

I prove there exists no assignment mechanism that is strategy-proof, non-wasteful and satisfies equal treatment of equals. When outside options may exist, this strengthens the impossibility result of Bogomolnaia and Moulin (2001) by weakening ordinal efficiency to non-wastefulness. My result solves an open question posed by Erdil (2014) and complements his results on the efficient frontier of random assignment mechanisms.

1. Introduction

In the assignment problem a number of heterogeneous, indivisible objects are to be distributed among several agents, with each agent entitled to at most one object.[1] There are no priorities and randomization is used to ensure fairness. Monetary transfers are disallowed. A *mechanism* elicits ordinal preferences of agents and outputs a random assignment of objects to agents.

I prove that if a mechanism is *strategy-proof* (truthfulness is a dominant strategy) and *fair* (equal treatment of equals: agents who report the same preferences face the same lottery over objects), then it is necessarily *wasteful*. A mechanism is wasteful if there exists an object x that is unassigned with positive probability and an agent who prefers x to another object (or the outside option) that she receives with positive probability.

Non-wastefulness is an ex-ante efficiency concept that is weaker than the standard notion of *ordinal efficiency* (not being first-order stochastically dominated). Bogomolnaia and Moulin (2001) prove that all strategy-proof and fair mechanisms are ordinally inefficient. My result strengthens theirs in the general setting where outside options may exist.

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[1] An important real-world application is housing assignment (public housing, campus housing, etc.). The assignment problem is also a building block of more complex matching problems, such as school choice (many-to-one matching, and priorities may be present) and course allocation (many-to-many matching).

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As an illustration, suppose there are four agents $i = 1, 2, 3, 4$ who report their true strict preferences $\succ^i$ over objects a, b, c and the outside option $\varnothing$. The canonical Random Serial Dictatorship (RSD) procedure[2] induces the *random assignment* shown center-left, a matrix whose row i shows the lottery over (a, b, c) faced by agent i.[3] Two other random assignments P_2 and P_3 are also shown.

Preferences	RSD (wasteful)			P_2 (non-wasteful)			P_3 (ordinally efficient)		
$a \succ^1 b \succ^1 c \succ^1 \varnothing$	5/12	1/12	5/12	5/12	1/12	1/2	1/2	0	1/2
$a \succ^2 b \succ^2 c \succ^2 \varnothing$	5/12	1/12	5/12	5/12	1/12	1/2	1/2	0	1/2
$b \succ^3 a \succ^3 \varnothing \succ^3 c$	1/12	5/12	0	1/12	5/12	0	0	1/2	0
$b \succ^4 a \succ^4 \varnothing \succ^4 c$	1/12	5/12	0	1/12	5/12	0	0	1/2	0

In the random assignment induced by RSD, object c is wasted: with probability 1/6 it is unassigned, yet agents 1 and 2 prefer c to receiving the outside option, which occurs with probability 1/12 each. P_2, where agents 1 and 2 receive c with probability 1/2, is a non-wasteful improvement over RSD. Still, it is ordinally inefficient: there are other random assignments, for example P_3, that first-order stochastically dominate it according to the true preferences.

The paper is organized as follows. In Section 2, I present the model. In Section 3, I state and prove the impossibility theorem. In Section 4, I verify minimality of the theorem's assumptions and discuss the importance of outside options. In Section 5, I conclude by discussing the relationship between waste and the set of undominated strategy-proof mechanisms.

2. Model

Let $N = \{1, 2, \ldots, n\}$ be a set of agents and $O = \{a, b, c, \ldots\}$ a set of m objects. Each agent $i \in N$ has strict preferences $\succ_i$ over O and the outside option $\varnothing$. Objects less preferred than the outside option are said to be *unacceptable*. Preferences $a \succ_i b \succ_i \varnothing \succ_i c$ (for example) will be represented compactly as a list $R^i = ab$; unacceptable objects are omitted from the list, as their ordering is irrelevant. $R = (R^i)_{i \in N}$ is the profile of preferences for all agents in N. Let $\mathcal{R}$ denote the set of all such possible profiles.

A *(random) assignment* is a matrix $P = (P_{ix})_{i \in N,\, x \in O}$, with rows indexed by agents $i \in N$ and columns indexed by objects $x \in O$. For each i and x, $P_{ix} \in [0, 1]$ is the probability that agent i receives object x. *Agent feasibility* holds if $\sum_{x \in O} P_{ix} \leq 1$ for each $i \in N$. *Object feasibility* holds if $\sum_{i \in N} P_{ix} \leq 1$ for each $x \in O$. P is *individually rational* with respect to preference profile $R = (R^i)_{i \in N}$ if $\varnothing \succ_i x$ implies $P_{ix} = 0$, for all $i \in N$ and $x \in O$.

A random assignment is *feasible* if it satisfies agent feasibility, object feasibility, and individual rationality. Let Π denote the set of all feasible random assignments. A generalization of the Birkhoff–von Neumann theorem (e.g. Kojima and Manea, 2010) ensures that all feasible random assignments can be decomposed as lotteries over deterministic assignments.

With respect to preference profile $R = (R^i)_{i \in N}$, a random assignment P:

- is *fair* or satisfies *equal treatment of equals* if $R^i = R^j$ implies $P_{ix} = P_{jx}$ for all $x \in O$;
- is *wasteful* if there exist $i \in N$, $x \in O$, and $y \in O \cup \{\varnothing\}$ such that $x \succ_i y$, $\sum_{i \in N} P_{ix} < 1$ and $P_{iy} > 0$. In words, x is wasted if it is unassigned with positive probability and there is an agent i who prefers it to an object (or the outside option) y that she receives with positive probability. P is *non-wasteful* if it is not wasteful;
- is *ordinally inefficient* if there exists another assignment $P' \neq P$ such that for all $i \in N$, the lottery over objects $(P'_{ix})_{x \in O}$ first-order stochastically dominates $(P_{ix})_{x \in O}$ according to R. In this case, we say that P' *dominates* P. If there is no such P', then P is *ordinally efficient*.

Ordinal efficiency implies non-wastefulness. To see this, let assignment P be wasteful, say at (i, x, y) as above. Then it is ordinally inefficient, because it is dominated by the assignment P' that takes P and moves probability mass $\min\{P_{iy}, 1 - \sum_{i \in N} P_{ix}\}$ from P_{iy} to P_{ix}.

An *(assignment) mechanism* is a function $P : \mathcal{R} \to \Pi$ that maps preference profiles $R \in \mathcal{R}$ into random assignments $P(R) \in \Pi$. A mechanism is individually rational; satisfies equal treatment of equals; is non-wasteful; is ordinally efficient, if for every $R \in \mathcal{R}$, $P(R)$ has that property.

A mechanism P is *strategy-proof* if for every agent i with preferences $\succcurlyeq^i$ represented by preference list R^i, every profile $R = (R^i, R^{-i})$, and every profile $R' = (\hat{R}^i, R^{-i})$ where i deviates to $\hat{R}^i$, the allocation (lottery over objects) that i receives at R first-order stochastically dominates the allocation at R' according to the true preferences $\succcurlyeq^i$. That is, for every object y

[2] Randomly choose an ordering over agents; in this order, assign agents to their preferred object among those that have not yet been assigned. RSD is strategy-proof, fair, and ex-post efficient (Abdulkadiroğlu and Sönmez, 1998).

[3] This approach was first employed by Hylland and Zeckhauser (1979), who show that it is always possible to decompose a random assignment as a lottery over deterministic assignments. Random assignment matrices are particularly well-suited to the strategic and ex-ante welfare analyses carried out in this paper.