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Note

Nash-implementation of the no-envy solution on symmetric domains of economies ☆

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ABSTRACT

We show that a simple game form, which resembles the “Divide-and-Choose” procedure, Nash-implements the no-envy solution on domains of economies where the set of feasible allocations is symmetric (an allocation obtained from a feasible allocation by interchanging the bundles of any two agents is also feasible) and preferences are complete (each agent can compare any two bundles). Our result extends a result by Thomson (2005) and it is applicable to a wide class of models including the classical model of fair allocation, the unidimensional single-peaked model, cake division model, and allocation of indivisible objects with monetary transfers. We show that, even when the preferences exhibit consumption externalities, an extension of the no-envy solution is Nash-implementable on general domains of economies.

1. Introduction

Our aim is to allocate a bundle of resources among agents equipped with preferences over their possible bundles. We want this allocation to be *envy-free*: no agent prefers some other agent's bundle to her own bundle (Foley, 1967). The no-envy solution selects the *envy-free* allocations for each economy.

No-envy is a central notion in the theory of fair allocation due to its intuitive appeal and wide applicability. A direct way of achieving this solution is to ask agents their preferences, and choose the *envy-free* allocations for the reported preferences. However, it is well known that it is not in general possible to truthfully elicit the information about preferences necessary to obtain *envy-free* allocations. For instance, for the model of allocating objects and a fixed amount of money, no selection from the no-envy solution is strategy-proof (Green and Laffont, 1979). Hence, we will consider Nash-implementation via game forms. A game form consists of a strategy set for each agent and a function specifying an outcome for each possible strategy profile. A game form Nash-implements a solution if for each profile of preferences, the set of Nash-equilibrium outcomes coincides with the set of allocations chosen by the solution for that profile. We show that the no-envy solution is Nash-implementable on symmetric domains of economies (an allocation obtained from a feasible allocation by interchanging the bundles of any two agents is also feasible), including the standard domains which have been studied before.[1] Moreover, we show that no-envy solution can be Nash-implemented by a simple game form.

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[1] Examples of studies that show Nash-implementability of the no-envy solution on standard domains are Doghmi (2013), Thomson (2005, 2010).

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A well-known procedure to achieve the no-envy solution when there are only two agents is "Divide-and-Choose". One agent divides the resource into two parts, and the other chooses one of those parts. For the problem of allocating some infinitely divisible goods among agents with strictly monotone preferences, Thomson (2005) proposes a game form, "Divide-and-Permute", which resembles Divide-and-Choose. But unlike Divide-and-Choose, it is not restricted to two-person problems. In Divide-and-Permute, two agents are "dividers". Each of them proposes an allocation, and they and everyone else announces a permutation. The outcome function is specified so that, whenever the dividers agree on the allocation, each agent can achieve each of its components. If the dividers don't agree, they are penalized by receiving zero. Despite its intuitive appeal, Divide-and-Permute is only applicable to models where there are two agents each of whom always prefers any bundle to a specific bundle (in Thomson, 2005, that least-preferred bundle is receiving nothing, since preferences are assumed to be strictly monotone). Existence of such least-preferred bundles enables the designer to punish the dividers when they do not agree on an allocation. However, there are allocation problems where such information is not available to the designer. For instance, for the model of allocating some objects and a fixed amount of money, an agent's least-preferred bundles at two different preference profiles may be different.

Here, we don't make any assumptions on the structure of the space to which the resource belong and on which preferences are admissible. The resource may be an infinitely divisible good, a finite collection of objects, an infinite collection of objects, or a collection of indivisible and infinitely divisible objects. The only assumptions we make are that preferences are complete, i.e. agents can compare any two bundles, and that the set of feasible allocations is symmetric, in the sense that if an allocation is feasible, then each allocation obtained from it by interchanging the bundles of any two agents is also feasible.

We show that, on any symmetric domain where the no-envy solution is nonempty valued, if there are at least three agents, a simple modification of Divide-and-Permute, which we call "Divide-and-Transpose", Nash-implements the no-envy solution. In our game form, which also resembles Divide-and-Choose, three agents are dividers, say agents $1, 2$, and 3. Each divider proposes an allocation, and each agent proposes a transposition of the bundles assigned to two agents (these two agents can be anyone including the proposer himself, and also they can be the same agent). The outcome function is specified so that, whenever a majority of the dividers agree on an allocation, each agent can achieve each component of that allocation. Otherwise, the allocation proposed by divider 1 is chosen, and each agent can achieve each component of that allocation. The simplicity of our game form comes from the fact that the strategies have straightforward economic interpretations. Moreover, at equilibrium, agents receive the bundles that have been announced. This corresponds to the truth-telling requirement in Dutta et al. (1995), which they consider as a simplicity indicator.[2]

One limitation of our result is that it is only applicable when there are at least 3 agents. However, as we discuss in Section 5, when there are only two agents, the no-envy solution is not anymore Nash-implementable on symmetric domains of economies. Moore and Repullo (1990) show that the existence of least-preferred bundles, which they call a "bad outcome", enables one to achieve positive implementation results for the two-agent case. Thus, implementability of the no-envy solution for the two-agent case in Thomson (2005) should be attributed to the existence of least-preferred bundles, which is not guaranteed in our model.

Another contribution of this paper is that, in Section 4, we allow preferences to exhibit consumption externalities, and show that a modified version of our game form Nash-implements an extension of the no-envy solution to that model (due to Velez, 2014). To our knowledge, this is the first paper to discuss Nash-implementation of the no-envy solution when there are consumption externalities. In Section 6, we discuss some of the allocation models for which the no-envy solution is well-defined, and to which our results apply.

2. The model

There is a bundle of resources to be distributed among a finite set of agents, denoted by $N = \{1, \ldots, n\}$, $n \geq 3$. Let Z be a subset of the Cartesian product of n sets and denote the set of feasible allocations with generic element $z = (z_1, \ldots, z_n)$. We assume that the set of feasible allocations is symmetric, that is, if an allocation z is feasible, then each allocation obtained from z by interchanging the bundles of any two agents is also feasible.[3] Each $i \in N$ is equipped with a preference relation over his possible consumption bundles. The only assumption we make on R_i is that it is complete. That is, for each pair of bundles (z_i, z_i'), $z_i \; R_i \; z_i'$ or $z_i' \; R_i \; z_i$. Let $\mathcal{R}_i$ denote the set of all such preferences. Let P_i denote the strict preference relation associated with R_i. An **economy** is a preference profile $R = (R_1, \ldots, R_n) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. Let $\overline{\mathcal{R}}$ denote the set of all economies, and $\mathcal{R} \subseteq \overline{\mathcal{R}}$ denote a set of admissible economies.

A **solution** is a correspondence $\varphi : \mathcal{R} \twoheadrightarrow Z$ associating with each economy a nonempty set of feasible allocations.

A **game form** is a pair $\Gamma = (S, g)$, where $S = S_1 \times \cdots \times S_n$ is a product of **strategy spaces**, and $g : S \to Z$ is an **outcome function**. Given $R \in \mathcal{R}$, let $\boldsymbol{E(\Gamma, R) \subseteq S}$ denote the set of Nash-equilibria of the game (Γ, R). Let $g(E(\Gamma, R)) = \{z \in Z | \exists s \in$

[2] For a more detailed discussion of the literature on what constitutes a simple game form, among others, see Dutta et al. (1995), Saijo et al. (1996), and Thomson (2005).

[3] In Section 3.1, we present some interesting allocation problems where the set of feasible allocations is not symmetric and discuss whether our results are still valid.