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Note

Imitative dynamics for games with continuous strategy space ☆

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ABSTRACT

This paper studies imitative dynamics for games with continuous strategy space. We define imitative dynamics—which include the replicator dynamic as a special case—as evolutionary dynamics that satisfy the imitative property and payoff monotonicity. Our definition of payoff monotonicity, which we use Radon–Nikodym derivatives to define, is weaker than the one proposed in Oechssler and Riedel (2002). We find that Oechssler and Riedel (2002)'s definition is too strong, and our definition is more adequate than theirs. We show that for a broad class of payoff functional dynamics, payoff monotonicity à la Oechssler and Riedel (2002) is equivalent to aggregate monotonicity in the sense of Samuelson and Zhang (1992). We then provide sufficient conditions for imitative dynamics and general evolutionary dynamics to be well-defined. Finally, with our definition of payoff monotonicity, a number of results that are standard for finite games extend to the case of games with continuous strategy space.

1. Introduction

There are recent studies of evolutionary dynamics for games with continuous strategy sets. The replicator dynamic, which is the best known evolutionary dynamic, has been studied by Bomze (1990, 1991), Oechssler and Riedel (2001, 2002), Cressman (2005), Cressman and Hofbauer (2005), and Cressman et al. (2006). Other evolutionary dynamics have also been studied. For example, the BNN dynamic has been studied by Hofbauer et al. (2009), pairwise comparison dynamics have been studied by Cheung (2014), and logit dynamics have been studied by Lahkar and Riedel (2015).

In this paper, we introduce a class of dynamics which generalize the replicator dynamic for games with continuous strategy space. We call them *imitative dynamics*. In the finite strategy case, a number of authors have introduced classes of imitative dynamics that generalize the replicator dynamic, for example, Nachbar (1990), Friedman (1991), Samuelson and Zhang (1992), Björnerstedt and Weibull (1996), Weibull (1995), Hofbauer (1995), and Ritzberger and Weibull (1995). Our generalization in the continuous strategy setting corresponds to *payoff monotonic dynamics* (cf. Weibull, 1995, Definition 4.2)[1] from the finite strategy setting.

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[1] The *payoff monotonicity* condition in the finite strategy setting has appeared in many places under different names, e.g., *relative monotonicity* in Nachbar (1990), *order compatibility of predynamics* in Friedman (1991), *monotonicity* in Samuelson and Zhang (1992), and *monotone percentage growth rates* in Sandholm (2010).

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We define imitative dynamics as evolutionary dynamics that satisfy the *imitative property*[2] and *payoff monotonicity*, which requires that strategies with higher payoffs have higher growth rates. Under this definition, the replicator dynamic is a special case of imitative dynamics.

Our definition of payoff monotonicity is different from the one proposed in Oechssler and Riedel (2002). Under their definition, a dynamic is payoff monotonic if *sets of strategies* with higher average payoffs have higher average growth rates. Our definition instead concerns the growth rate of each individual strategy by using Radon–Nikodym derivatives. We find that Oechssler and Riedel (2002)'s definition is not satisfactory since it is too strong and rules out certain basic examples whose finite strategy counterparts satisfy payoff monotonicity for the finite strategy case (see Example 3). The reason behind is that their condition is an "averaging" condition. We show that for a broad class of payoff functional dynamics, payoff monotonicity à la Oechssler and Riedel (2002) is equivalent to aggregate monotonicity in the sense of Samuelson and Zhang (1992).

We then study conditions under which a general evolutionary dynamic is well-defined, i.e., solutions for the dynamic exist and are unique. We find that the condition in Theorem 1 of Cheung (2014), which is sufficient for a pairwise comparison dynamic to be well-defined, is also sufficient for an imitative dynamic to be well-defined. Moreover, the condition is sufficient for any general evolutionary dynamic that is derived from the mean dynamic to be well-defined, no matter the dynamic is imitative or direct.

Finally, we show that with our definition of payoff monotonicity, a number of results that are standard for finite games extend to the case of games with continuous strategy space. For any imitative dynamic, positive correlation is satisfied, the rest points coincide with the restricted equilibria,[3] and Lyapunov stability implies Nash equilibrium (Propositions 1–3). Combining these results with the results in Cheung (2014), we obtain global convergence and local stability results for imitative dynamics in potential games.

2. Settings

2.1. Population games

Let $\mathcal{S}$ be a compact metric space with metric d. Our main interest is in cases where $\mathcal{S}$ is a continuum, for instance a compact convex set of $\mathbb{R}^n$. But since we only require compactness, $\mathcal{S}$ may also be a finite set, or a union of finite and continuous sets. Thus our results here generalize standard results for the finite strategy case from the literature.

Consider a unit mass of agents, each of whom chooses a pure strategy from $\mathcal{S}$. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathcal{S}$. Denote by $\mathcal{M}_1^+(\mathcal{S})$ the space of probability measures on $(\mathcal{S}, \mathcal{B})$, and by $\mathcal{M}(\mathcal{S})$ the space of finite signed measures. Then $\mathcal{M}(\mathcal{S})$ is a vector space and is the linear span of $\mathcal{M}_1^+(\mathcal{S})$. A *population state* is a distribution over strategies and is described by a probability measure $\mu \in \mathcal{M}_1^+(\mathcal{S})$.

We identify a *population game* with a map

$$F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$$

that is continuous with respect to the weak topology, where $\mathcal{C}_b(\mathcal{S})$ is the space of continuous (and hence bounded) functions on $\mathcal{S}$ with the supremum norm. The weak topology is related to weak convergence of measures. A sequence of measures $\mu_n \in \mathcal{M}(\mathcal{S})$ converges weakly to $\mu \in \mathcal{M}(\mathcal{S})$, written $\mu_n \xrightarrow{w} \mu$, if $\int_{\mathcal{S}} f\, d\mu_n \to \int_{\mathcal{S}} f\, d\mu$ for all $f \in \mathcal{C}_b(\mathcal{S})$. The weak topology on $\mathcal{M}(\mathcal{S})$ is the coarsest topology (i.e., the topology with the fewest open sets) on $\mathcal{M}(\mathcal{S})$ such that $\mu \mapsto \int_{\mathcal{S}} f\, d\mu$ is continuous for all $f \in \mathcal{C}_b(\mathcal{S})$.[4] A map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is continuous with respect to the weak topology if $F(\mu_n) \to F(\mu)$ (in the supremum norm) for any sequence $\{\mu_n\} \subseteq \mathcal{M}_1^+(\mathcal{S})$ such that $\mu_n \xrightarrow{w} \mu$. We may call such a map F *weakly continuous*.[5]

We denote by $F_x(\mu)$ the payoff of pure strategy $x \in \mathcal{S}$ at population state $\mu \in \mathcal{M}_1^+(\mathcal{S})$, and $F(\mu)$ specifies payoffs at all strategies in $\mathcal{S}$ at state μ. We call $F(\mu)$ the *payoff profile* at state μ. The *population-weighted average payoff* (or the *aggregate payoff*) obtained by the unit mass of agents at state $\mu \in \mathcal{M}_1^+(\mathcal{S})$ is

$$\bar{F}(\mu) = \int_{\mathcal{S}} F_x(\mu)\, \mu(dx).$$

[2] A dynamic is *imitative*, in contrast to *direct* (or *innovative*), means that under the dynamic, when an agent receives an opportunity to switch strategies, he chooses a candidate strategy at random according to the distribution of strategies in the population. This is usually interpreted as that the revising agent randomly chooses an opponent from the population and *imitates* the opponent with a probability depending on the revision protocol.

[3] A restricted equilibrium of a population game is a Nash equilibrium of the game in which only strategies in the support of the restricted equilibrium can be played. In particular, all Nash equilibria are restricted equilibria.

[4] See, e.g., Ekeland and Témam, 1999, pp. 6.

[5] A common example of a population game is one generated by pairwise matching, i.e., F is defined by $F_x(\mu) := \int_{\mathcal{S}} h(x, y)\, \mu(dy)$ for $\mu \in \mathcal{M}_1^+(\mathcal{S})$ and $x \in \mathcal{S}$, where $h : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is the single match payoff function. If h is continuous (and hence bounded), then F is a weakly continuous map from $\mathcal{M}_1^+(\mathcal{S})$ to $\mathcal{C}_b(\mathcal{S})$ and so $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defines a population game.