Games and Economic Behavior 95 (2016) 107–112

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Note

An ordinal minimax theorem

Felix Brandt [a,*], Markus Brill [b], Warut Suksompong [c]

[a] Institut für Informatik, TU München, Germany
[b] Department of Computer Science, Duke University, USA
[c] Department of Computer Science, Stanford University, USA

ARTICLE INFO

Article history:
Received 26 May 2015
Available online 11 January 2016

JEL classification:
C72

Keywords:
Zero-sum games
Shapley
Saddles
Minimax theorem

ABSTRACT

In the early 1950s Lloyd Shapley proposed an ordinal and set-valued solution concept for zero-sum games called *weak saddle*. We show that all weak saddles of a given zero-sum game are interchangeable and equivalent. As a consequence, every such game possesses a unique set-based value.

1. Introduction

One of the earliest solution concepts considered in game theory are *saddle points*, combinations of actions such that no player can gain by deviating (see, e.g., von Neumann and Morgenstern, 1947). In two-player zero-sum games, every saddle point happens to coincide with the optimal outcome both players can guarantee in the worst case and thus enjoys a very strong normative foundation. Unfortunately, however, saddle points are not guaranteed to exist. This situation can be rectified by the introduction of *mixed*—i.e., randomized—strategies, as first proposed by Borel (1921). Von Neumann (1928) proved that every zero-sum game contains a mixed saddle point, or equilibrium. While equilibria need not be unique, they maintain two appealing properties of saddle points: *interchangeability* (any combination of equilibrium strategies for either player forms an equilibrium) and *equivalence* (all equilibria yield the same expected payoff).

Mixed equilibria have been criticized for resting on demanding epistemic assumptions such as the expected utility axioms by von Neumann and Morgenstern (1947). See, for example, Luce and Raiffa (1957, pp. 74–76) and Fishburn (1978). As Aumann puts it: "When randomized strategies are used in a strategic game, payoff must be replaced by expected payoff. Since the game is played only once, the law of large numbers does not apply, so it is not clear why a player would be interested specifically in the mathematical expectation of his payoff" (Aumann, 1987, p. 63).

Shapley (1953a, 1953b) showed that the existence of saddle points can also be guaranteed by moving to *minimal sets* of actions rather than randomizations over them.[1] Shapley defines a *generalized saddle point (GSP)* to be a tuple of subsets of actions for each player that satisfies a simple external stability condition: Every action not contained in a player's subset is dominated by some action in the set, given that the other player chooses actions from his set. A GSP is minimal if it does

* Corresponding author.
E-mail address: brandtf@in.tum.de (F. Brandt).

[1] The main results of the 1953 reports later reappeared in revised form (Shapley, 1964).

http://dx.doi.org/10.1016/j.geb.2015.12.010
0899-8256/

$$A_1 = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ 0 & 3 & 4 & 4 & 1 \\ 0 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 2 & 2 & 1 & 3 & 2 \\ 2 & 4 & 0 & 0 & 2 \\ 1 & 3 & 3 & 4 & 1 \\ 2 & 3 & 1 & 3 & 2 \\ 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

Fig. 1. Three example zero-sum games. For each game, the rows and columns are labeled $r_1, r_2, \ldots$ and $c_1, c_2, \ldots$, respectively. The game A_1 contains one weak saddle: $\{r_1, r_2\} \times \{c_1, c_2, c_3\}$. The game A_2 contains a saddle point $\{r_1\} \times \{c_1\}$. This saddle point is the unique pure Nash equilibrium and the unique weak saddle of this game. Moreover, $(\frac{1}{2}r_2 + \frac{1}{2}r_3, \frac{1}{2}c_2 + \frac{1}{2}c_3)$ is a (mixed) Nash equilibrium of A_2. The game A_3 contains four weak saddles: $\{r_1, r_3\} \times \{c_1, c_3\}$, $\{r_1, r_3\} \times \{c_3, c_5\}$, $\{r_3, r_4\} \times \{c_1, c_3\}$, and $\{r_3, r_4\} \times \{c_3, c_5\}$. For all three games, the product of all rows and all columns is the unique strict saddle.

not contain another GSP. Minimal GSPs, which Shapley calls *saddles*, also satisfy internal stability in the sense that no two actions within a set dominate each other, given that the other player chooses actions from his set. While Shapley was the first to conceive GSPs, he was not the only one. Apparently unaware of Shapley's work, Samuelson (1992) uses the very related concept of a *consistent pair* to point out epistemic inconsistencies in the concept of iterated weak dominance. Also, *weakly admissible sets* as defined by McKelvey and Ordeshook (1976) in the context of spatial voting games and the *minimal covering set* as defined by Dutta (1988) in the context of majority tournaments are GSPs (Duggan and Le Breton, 1996a).[2]

In this paper, we consider GSPs with respect to weak dominance. An action weakly dominates another action if it always yields at least as much utility. Shapley (1964, p. 10) notes that no general uniqueness result is available for this type of saddle. Later, uniqueness has been shown for restricted classes of zero-sum games, namely tournament games (Dutta, 1988) and confrontation games (Duggan and Le Breton, 1996a). We show that all weak saddles of a given zero-sum game are interchangeable and equivalent. This implies the above-mentioned uniqueness results and shows that every zero-sum game possesses a unique set-based value. Our result can be interpreted as an ordinal variant of the minimax theorem.

2. Preliminaries

A finite two-player zero-sum game is given by a matrix $A = (a_{i,j})_{i \in R, j \in C}$. The finite set R of rows represents the row player's actions, and the finite set C of columns represents the column player's actions. If the row player chooses action $r \in R$, and the column player chooses action $c \in C$, then the *payoff* (or *utility*) of the row player is given by the entry $a_{r,c}$ of the matrix, while the payoff of the column player is given by $-a_{r,c}$. For nonempty subsets $R' \subseteq R$ and $C' \subseteq C$, $A|_{R' \times C'}$ denotes the *subgame* in which the row player has action set R' and the column player has action set C'.

An action $r_1 \in R$ *weakly dominates* another action $r_2 \in R$ with respect to a set $C' \subseteq C$ of columns, denoted $r_1 \geq_{C'} r_2$, if $a_{r_1,c} \geq a_{r_2,c}$ for all $c \in C'$.[3] Similarly, an action $c_1 \in C$ *weakly dominates* another action $c_2 \in C$ with respect to a set $R' \subseteq R$ of rows, denoted $c_1 \leq_{R'} c_2$, if $-a_{r,c_1} \geq -a_{r,c_2}$ (and thus $a_{r,c_1} \leq a_{r,c_2}$) for all $r \in R'$. *Strict dominance* is defined analogously, with the weak inequalities replaced by strict inequalities.

Dominance relations can be extended to sets of actions as follows. A set R_1 of rows *weakly* (resp. *strictly*) *dominates* a set R_2 of rows with respect to $C' \subseteq C$ if for every row $r_2 \in R_2$, there exists a row $r_1 \in R_1$ such that r_1 weakly (resp. strictly) dominates r_2 with respect to C'. We denote this by $R_1 \geq_{C'} R_2$ (resp. $R_1 >_{C'} R_2$). Dominance between sets of columns is defined analogously, and denoted $C_1 \leq_{R'} C_2$ (for weak dominance) and $C_1 <_{R'} C_2$ (for strict dominance).

We are now prepared to define *saddles*, which are based on the notion of a *generalized saddle point (GSP)* (Shapley 1953a, 1953b, 1964). Given a subset $R' \subseteq R$ of rows and a subset $C' \subseteq C$ of columns, the product $R' \times C'$ is a *weak GSP* if $R' \geq_{C'} R \backslash R'$ and $C' \leq_{R'} C \backslash C'$. Furthermore, the product $R' \times C'$ is a *weak saddle* if it is a weak GSP and no proper subset of it is a weak GSP.[4] Strict GSPs and strict saddles are defined analogously.

In contrast to strict saddles, weak saddles are extensions of saddle points in the sense that every saddle point constitutes a weak saddle. Since the product $R \times C$ containing all actions is a trivial weak and strict GSP of any game, weak and strict saddles are guaranteed to exist. While strict saddles have been shown to be unique in zero-sum games (see Corollary 3), this is not the case for weak saddles. It is noteworthy that saddles generally *cannot* be found by the iterated elimination of (weakly or strictly) dominated actions.[5] See Fig. 1 for examples.

3. The result

In this section, we prove that weak saddles in zero-sum games are interchangeable and equivalent. We begin with a lemma.

[2] GSPs have also been considered in the context of general normal-form games (see, e.g., Duggan and Le Breton, 1996b; Brandt et al., 2009, 2011; Brandt and Brill, 2012).

[3] What we call weak dominance here is sometimes also called *very weak dominance* (see, e.g., Leyton-Brown and Shoham, 2008).

[4] Weak saddles have been called *very weak saddles* by Brandt et al. (2011); see also Footnote 3. In some papers (e.g., Duggan and Le Breton 1996a, 2001; Brandt et al., 2009, 2011), the dominance used for weak saddles requires at least one strict inequality. In the context of confrontation games (see Corollary 4), where weak saddles have usually been considered, both notions of weak saddles coincide. Shapley (1953a, 1953b, 1964) defines weak saddles as we do here. It is easily seen that our theorem does not hold for weak saddles that require at least one strict inequality (see, for example, the restriction to the first two rows and columns of game A_2 in Fig. 1).

[5] While the subgames generated by iteratively eliminating dominated strategies are GSPs, these GSPs need not be minimal.