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How to gamble against all odds

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A B S T R A C T

We compare the power of betting strategies (aka *martingales*) whose wagers take values in different sets of reals. A martingale whose wagers take values in a set A is called an A-martingale. A set of reals B *anticipates* a set A, if for every A-martingale there is a countable set of B-martingales, such that on every binary sequence on which the A-martingale gains an infinite amount at least one of the B-martingales gains an infinite amount, too.

We show that for two important classes of pairs of sets A and B, B anticipates A if and only if the closure of B contains rA, for some positive r. One class is when A is bounded and B is bounded away from zero; the other class is when B is well ordered. Our results generalize several recent results in algorithmic randomness and answer a question posed by Chalcraft et al. (2012).

1. Introduction

Randomness and computation are related to two basic elements of bounded rationality modeling: (a) players cannot generate truly random actions; (b) they cannot implement arbitrarily complex strategies. When rationality is computationally constrained, true randomness can be replaced by pseudo-randomness: one need not generate truly random actions, just actions whose pattern is too complicated for any rationally bounded agent to be able to recognize.

Algorithmic randomness is the field of computer science that studies the relations between computability and randomness. It provides the rigorous definitions and analytic tools to address questions involving the tension between the two above elements of bounded rationality. On the other hand, the present paper applies game theory to solve a problem from algorithmic randomness by recasting the problem as an extensive form game.

1.1. Gambling house game

Gambler 0, the cousin of the casino owner, repeatedly bets on red/black. She is only allowed to wager even (positive) integers, i.e., she can bet 4 dollars on black, then 10 on red, then 2 on black, etc. ("betting 4 on black" means that you gain 4 if black occurred, and lose 4 if red did). Then the *regular* gamblers make their bets, and they are only allowed to wager odd integers. The red/black outcome is not decided by a lottery; rather, the casino owner chooses red or black. He attempts to aid gambler 0 by his choices.[2]

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[2] Following the standard algorithmic randomness formalism, it is assumed that the gamblers (particularly, Gambler 0) do not observe each other's bets or gains, but only observe the casino sequence of reds and blacks. This assumption turns out to be immaterial for our results.

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All gamblers start with some finite, not necessary equal, wealth. Going into debt is not allowed, i.e., you cannot bet more than you have. Say that a gambler "succeeds" if, along the infinite stream of bets, her wealth tends to infinity. The "home team", consisting of the casino owner and gambler 0, wins if and only if gambler 0 succeeds while the other gamblers do not.

In the case depicted above, gambler 0 uses even integers as wagers, and the others use odd wagers. We will see that the home team can guarantee a win in this case and that they could not had it been the other way round, namely, odd integers for gambler 0 and even integers for the others. In general, let A be the wagers allowed to gambler 0, and B those allowed to the other gamblers (A, B being any pair of subsets of the positive real numbers). We ask for which A and B the home team can *guarantee a win*, in the following sense: gambler 0 can choose a strategy that guarantees that for any strategies that the other gamblers choose the casino can choose a sequence so that the home team wins.

As a simple example, suppose that apart from gambler 0 there is only one more, regular, gambler (call him gambler 1), and first, let $A = \{2\}$ and $B = \{1\}$. Then the home team cannot guarantee a win: for any pure strategy of gambler 0, let gambler 1 employ the strategy that is a copy of it, only wagering 1 instead of 2. Then the gains (or losses) of 1 are exactly half the gains of 0. Therefore, if gambler 0 succeeds, so does gambler 1. Second, let B be as before, and let $A = \{1, 2\}$. Then the home team can guarantee a win as follows. The strategy of gambler 0 is simple: she always bets on red; at the start she wagers two dollars, and keeps doing this as long as she wins. Then, after the first loss, she switches to wagering one dollar ad infinitum. The casino strategy chooses, in a first phase, red every time, until the wealth of 0 exceeds 1's wealth by more than three dollars (note that this is bound to eventually happen, since in the first phase 0 gains two every time, while 1 gains at most one). Then it chooses black once (after which the wealth of 0 still exceeds the wealth of 1, as 0 lost two dollars and 1 gained no more than one dollar). Then comes the second phase, where the casino always chooses the opposite color to what gambler 1 bet on, and chooses red if 1 did not bet anything. This strategy of the casino ensures that, during the second phase, 1 cannot make a non-zero bet more than a finite number of times before he goes broke (and if 1 does go broke, 0 still does not). Hence, whatever strategy he chooses, 1 will not succeed, while 0 will.

1.2. Computability, randomness and unpredictability

This casino game is related to the notions of randomness and computability.[3] In the theory of *algorithmic randomness*, a sequence of zeros and ones is called *computably random* if there exists no computable strategy that succeeds against it (thus, the sequence is in some sense unpredictable). For a set A of positive real numbers, a sequence is *A-valued random* if there exists no computable gambling strategy (aka "martingale"), with wagers in A, that succeeds against it. Hence we may compare two sets of reals A and B, and ask whether B-valued randomness implies A-valued randomness. In other words, whether for any casino sequence x, and a computable A-martingale (gambling strategy with wagers in A) that succeeds on x, there exists a B-martingale that succeeds on x. And in terms of our casino game, suppose that *every* computable B-martingale is employed by some regular gambler. Then the question is whether there exist a computable A-martingale and a casino sequence, such that the home team wins.

There are countably many computable strategies. In many cases this countability, rather than the specific type of admissible strategies, is the essential point in the analysis. This naturally leads to the following setting (Definition 2): gambler 0's wagers are in A, and there are countably many regular players, whose wagers are in B. We will say that *A evades B* if the home team can guarantee a win. This is the central notion in our paper. Thus, we are not directly concerned with computability, or other complexity considerations,[4] but rather take a more abstract view: we only require that the number of strategies against which the home team needs to concurrently combat is countable (see also Remark 1 below). One may consider a few variants of this settings (see below).

If A does not evade B, we say that *B anticipates A*. If B contains A, then clearly B anticipates A, because the strategy of gambler 1 can be an exact copy of 0's strategy (and there is even no need for countably many regular gamblers, one is enough). Therefore, if 0 succeeds, so does 1. Similarly, if B contains a multiple of A, i.e., $B \supseteq rA$ for some $r > 0$, then B anticipates A: as in the example above, the B-strategy can be the same as the A-strategy up to the multiple r (i.e., the wagers are multiplied by r), and therefore the gains are the same, multiplied by r. For example, the even integers anticipate the odd integers (or the whole set of integers, for that matter).

Thus, B containing a multiple of A is a sufficient condition for B to anticipate A. Is it also a necessary condition? Chalcraft et al. (2012) showed that if A and B are finite, then it is necessary. They asked whether this characterization extends to infinite A and B (note that their framework is that of algorithmic randomness, i.e., it only allows for computable martingales). In other words:

($*$) Does B not containing a multiple of A imply that A evades B?

[3] For recent examples of computability considerations directly applied to game-theoretic settings, see Hu (2014), and to forecasting, see V'yugin (2009) and Hu and Shmaya (2013).

[4] Rod Downey (private communication) pointed out that in the case where A and B are recursive sets of rational numbers our analysis applies to the question whether every B-valued random is A-valued random.