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Monotone equilibria in nonatomic supermodular games. A comment ☆

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ABSTRACT

Recently Yang and Qi (2013) stated an interesting theorem on the existence of complete lattice of equilibria in a particular class of large nonatomic supermodular games for general action and players spaces. Unfortunately, their result is incorrect. In this note, we detail the nature of the problem with the stated theorem, provide a counterexample, and then correct the result under additional assumptions.

1. Introduction

In an interesting recent paper by Yang and Qi (2013), the authors study the existence of monotone pure-strategy Nash equilibrium in a semi-anonymous nonatomic supermodular game. In contrast to the recent work by Balbus et al. (2014, 2015), where complementarity assumptions for player payoffs are assumed between their individual *actions*, in Yang and Qi (2013) the authors additionally assume complementarities between individual actions and *names* (that could also be interpreted as players' traits). In their paper, the primary technical reason for assuming these additional sources of complementarities is it allows the authors to study Nash equilibria that are monotone in names/traits, which can potentially resolve some problems of measurability of pure-strategies. In particular, the authors assume the name/trait space is a complete chain, and then argue that monotone functions on such a name/trait space are Borel-measurable.[1] If this were true, this would imply that, under pointwise partial order, the space of monotone (and hence measurable) pure-strategies would form a complete lattice, a step that is critical for the application of the Veinott–Zhou's version of Tarski's theorem that the authors use to verify existence of a Nash equilibrium. Unfortunately, monotonicity of pure-strategies in this general setting does *not* imply their measurability.

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[1] A set T is said to be a *complete chain* if T is (a) a linearly ordered (i.e., T is a partially ordered set such that any two elements are comparable), and (b) complete (i.e., for any subset $T_1 \subset T$, $\sup T_1$ and $\inf T_1$ are both elements of T).

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So, in this note, we first show that under the assumptions in Yang and Qi (2013), there is not sufficient structure to verify existence of a Nash equilibrium. We do this by providing a counterexample to a key lemma in the paper needed to prove their main theorem. We then show, how under some additional assumptions, the situation can be remedied. We finally conclude by discussing, how the methods of Balbus et al. (2014, 2015) can be used to study the question of monotone equilibria in a class of large games with more general strategic complementarities (that include nonatomic supermodular games).

2. The environment

We follow the notation in Yang and Qi (2013) in this note. A semi-anonymous game with an ordered set of players' names/traits is a tuple $\Gamma = (T, p, S, f)$, where T is a partially ordered set (a poset) of players' names/traits, $p \in \mathcal{P}(T)$ is a probability distribution on the measurable space $(T, \mathcal{B}(T))$, where $\mathcal{B}(T)$ is the Borel σ-field generated by the interval topology on T, S is an action space of the players, $f : S \times \mathcal{P}(T \times S) \times T \to \mathbb{R}$ is a payoff function where $f(s, r, \theta)$ denotes the payoff for player $\theta \in T$, using action $s \in S$, while facing a joint trait-action distribution given by $r \in \mathcal{P}(T \times S)$.

Denote by $\mathcal{M}(T, S)$ the set of Borel measurable functions from T to S endowed with the pointwise partial order, and let i_T denote the identity mapping. Then, a Nash equilibrium of the game Γ is a function $x \in \mathcal{M}(T, S)$ that satisfies the following:

$$\forall \theta \in T, \forall s \in S \text{ we have } f(x(\theta), p \circ (i_T, x)^{-1}, \theta) \geq f(s, p \circ (i_T, x)^{-1}, \theta), \tag{1}$$

where $p \circ (i_T, x)^{-1}$ is a joint trait-action distribution on $T \times S$ implied by x.

As in Yang and Qi (2013), assume T is a complete chain and S a complete lattice. Endow $\mathcal{P}(T \times S)$ with the first stochastic dominance order. Yang and Qi (2013) impose the following assumptions on game Γ:

Assumption 1. Assume:

- $s \to f(s, r, \theta)$ is order upper semi-continuous for each $\theta \in T$ and $r \in \mathcal{P}(T \times S)$,
- $s \to f(s, r, \theta)$ is supermodular for each $\theta \in T$ and $r \in \mathcal{P}(T \times S)$,
- f has increasing differences with s and (θ, r).

The main existence result of Yang and Qi (2013) states that under Assumption 1, the set of monotone Nash equilibria in $\mathcal{M}(T, S)$ of the game Γ is a nonempty complete lattice. This result is obtained essentially as an application of the Veinott–Zhou fixed point theorem (e.g., see Veinott, 1992 or Zhou, 1994), which in turn is a generalization of the Tarski (1955) fixed point theorem to the case of strong set order ascending best response correspondences. By $I(T, S)$ denote the set of increasing mappings from T to S.

To prove their main theorem, Yang and Qi (2013) claim the following lemma is true:

Lemma 1. *(See Yang and Qi, 2013.) Suppose T is a complete chain and S is a complete lattice, then $I(T, S) \subset \mathcal{M}(T, S)$.*

This lemma is critical for the existence result in Yang and Qi (2013). It claims that *any* monotone increasing function on the complete chain is Borel-*measurable*. In fact, if this lemma was correct, it would imply the set of monotone and measurable functions, $I(T, S) \cap \mathcal{M}(T, S)$, is a complete lattice under pointwise partial orders. Therefore, if the best response correspondence transforms the space $I(T, S) \cap \mathcal{M}(T, S)$ into $I(T, S)$, it necessarily maps into $I(T, S) \cap \mathcal{M}(T, S)$.

Unfortunately, Lemma 1 is not correct, and as a consequence, the main existence theorem in the paper is incorrect as well. Specifically, and related to the issues raised in a recent paper by Balbus et al. (2014), not only is this approach to the proof of the existence in this game generally inappropriate; it is in fact wrong under their assumptions (as the best response correspondence can map *outside* the set of measurable functions).

In the remainder of the paper, we present a counterexample to Lemma 1 (in Theorem 1 in the next section). Then, we discuss sufficient conditions for existence of equilibrium in a version of the actual game considered in Yang and Qi (2013).

3. A counterexample

Consider an interval I of ordinal numbers,[2] and let ω_1 be the first uncountable ordinal number (the least number of the set $\{x \in I : \#([0, x]) \geq \aleph_1\}$). Clearly every initial segment of the subset $[0, \omega_1)$ is countable but $\Omega := [0, \omega_1]$ has cardinality $\aleph_1$.

Following Jech (2010), we define the order topology:

Definition 1. The order topology $\mathcal{O}$ is defined as the topology, whose subbase of closed sets is formed from all sets $F \subset \Omega$ satisfying the following condition:

[2] Our construction in this section is similar to that presented in Theorem 1.14, page 19 and Example 12.9, page 439 in Aliprantis and Border (2006).